3. (a) Let $\mathcal{M}$ be an infinite σ-algebra of subsets of some set X. There exists a countably infinite subcollection $\mathcal{C} \subseteq \mathcal{M}$, and we may choose $\mathcal{C}$ to be closed under taking complements (adding in missing complements if necessary). For each $x \in X$, define $D_x := \cap\{C \in \mathcal{C} \mid x \in C\}$, so that $D_x \in \mathcal{M}$. Let $x, y \in X$ and suppose $y \in D_x$. Then $y \in C$ for all $C \in \mathcal{C}$ with $x \in C$. Moreover, if $y \in C$ for some $C \in \mathcal{C}$, then $y \notin C^c \in \mathcal{C}$, so $x \notin C^c$ and hence $x \in C$. This implies that, if $C \in \mathcal{C}$, then $x \in C$ iff $y \in C$. In particular $D_x = D_y$. If $x, y \in X$ and there exists $z \in D_x \cap D_y$, then $D_x = D_z = D_y$, so the collection $\mathcal{D} := \{D_x \mid x \in X\}$ is pairwise disjoint. If $C \in \mathcal{C}$, then $C = \cup\{D_x \mid x \in C\}$, since $x \in D_x \subseteq C$ for all $x \in C$. Therefore, the image of the map $\cup : 2^{\mathcal{D}} \to 2^X$ contains $\mathcal{C}$, so there exists a surjection from some subset of $2^{\mathcal{D}}$ onto $\mathcal{C}$. If $\mathcal{D}$ were finite, then $2^{\mathcal{D}}$ would be as well, which is a contradiction because $\mathcal{C}$ is infinite. Therefore $\mathcal{D}$ is an infinite, pairwise disjoint subcollection of $\mathcal{M}$.

   (b) Let $\mathcal{M}$ be an infinite σ-algebra of subsets of some set X. By the previous exercise, there exists an infinite, pairwise disjoint subcollection $\mathcal{D}$ of $\mathcal{M}$. Since $\mathbb{Q}$ is countable, there exists an injection $q : \mathbb{Q} \to \mathcal{D}$. Define $r : 2^{\mathbb{Q}} \to \mathcal{M}$ by $r(A) := \cup_{a \in A} q(a)$. Then r is well-defined because each $A \subseteq \mathbb{Q}$ is countable, and injective because $\mathcal{D}$ is pairwise disjoint. There exists an injection from $\mathbb{R} \to 2^{\mathbb{Q}}$, for example the map $x \mapsto (-\infty, x] \cap \mathbb{Q}$ (which is injective by the least upper bound property of $\mathbb{R}$). Composing these injections gives an injective map from $\mathbb{R} \to \mathcal{M}$, which shows that $\text{card}(\mathcal{M}) \geq \mathfrak{c}$.

4. Let $\mathcal{A}$ be an algebra which is closed under countable increasing unions. To show that $\mathcal{A}$ is a σ-algebra, it suffices to show that it is closed under arbitrary countable unions. To this end, let $\{E_n\}_{n \in \mathbb{N}}$ be a countable collection of members of $\mathcal{A}$. For each $n \in \mathbb{N}$ define $F_n := \cup_{k=1}^n E_k$. Since $\mathcal{A}$ is an algebra it is clear that $(F_n)_{n \in \mathbb{N}}$ is an increasing sequence of members of $\mathcal{A}$, so $\cup_{n \in \mathbb{N}} E_n = \cup_{n \in \mathbb{N}} F_n \in \mathcal{A}$ as required. Conversely, it is plain that every σ-algebra is closed under countable increasing unions.

5. Let $\mathcal{E}$ be a collection of subsets of some set X, and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{E}$. Define

$$\mathcal{N} := \cup\{\mathcal{A} \mid \mathcal{A} \text{ is the } \sigma\text{-algebra generated by some countable subcollection of } \mathcal{E}\}.$$

It is clear that $\varnothing, X \in \mathcal{N}$, $\mathcal{E} \subseteq \mathcal{N}$ and $\mathcal{N}$ is closed under taking complements. Let $\{E_n\}_{n \in \mathbb{N}}$ be a countable collection of members of $\mathcal{N}$. For each $n \in \mathbb{N}$ there exists a countable subcollection $\mathcal{E}_n$ of $\mathcal{E}$ such that E_n is in the σ-algebra generated by $\mathcal{E}_n$. Since union of countably many countable sets is countable, $\mathcal{F} := \cup_{n \in \mathbb{N}} \mathcal{E}_n$ is a countable subcollection of $\mathcal{E}$. Let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{F}$, so that $\mathcal{A} \subseteq \mathcal{N}$. For each $n \in \mathbb{N}$ the σ-algebra generated by $\mathcal{E}_n$ is a subset of $\mathcal{A}$, because $\mathcal{A}$ is a σ-algebra containing $\mathcal{E}_n$. This implies that $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, and hence $\cup_{n \in \mathbb{N}} E_n \in \mathcal{A} \subseteq \mathcal{N}$. Therefore $\mathcal{N}$ is closed under countable unions, so it is a σ-algebra containing $\mathcal{E}$. This implies that $\mathcal{M} \subseteq \mathcal{N}$.

Conversely, let $E \in \mathcal{N}$. Then E belongs to the σ-algebra $\mathcal{A}$ generated by some countable subcollection of $\mathcal{E}$. Since $\mathcal{M}$ is a σ-algebra containing this subcollection, $\mathcal{A} \subseteq \mathcal{M}$ and hence $E \in \mathcal{M}$. This shows that $\mathcal{N} \subseteq \mathcal{M}$ and hence $\mathcal{N} = \mathcal{M}$.

6. Note that $\overline{\mu}(\varnothing) = \overline{\mu}(\varnothing \cup \varnothing) = \mu(\varnothing) = 0$. Given a sequence $(A_n)_{n \in \mathbb{N}}$ of disjoint sets in $\overline{\mathcal{M}}$, there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{M}$ and a sequence $(F_n)_{n \in \mathbb{N}}$ of subsets of measure zero sets from $\mathcal{M}$ such that $A_n = E_n \cup F_n$ for all $n \in \mathbb{N}$. Note that $(E_n)_{n \in \mathbb{N}}$ is pairwise disjoint, and $\cup_{n \in \mathbb{N}} F_n$ is a subset of a measure zero set. Therefore

$$\overline{\mu}(\cup_{n \in \mathbb{N}} A_n) = \overline{\mu}((\cup_{n \in \mathbb{N}} E_n) \cup (\cup_{n \in \mathbb{N}} F_n)) = \mu(\cup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \mu(E_n) = \sum_{n \in \mathbb{N}} \overline{\mu}(A_n).$$

This shows that $\overline{\mu}$ is a measure. Let $A \subseteq X$, and suppose there exists $B \in \overline{\mathcal{M}}$ such that $A \subseteq B$ and $\overline{\mu}(B) = 0$. Then $B = E \cup F$ for some $E \in \mathcal{M}$ and F a subset of a measure zero set $N \in \mathcal{M}$. It follows that $A \subseteq E \cup N$, where $E \cup N \in \mathcal{M}$ and $\mu(E \cup N) \leq \mu(E) + \mu(N) = \overline{\mu}(E) + 0 \leq \overline{\mu}(B) = 0$. Therefore $A = \varnothing \cup A \in \overline{\mathcal{M}}$, which shows that $\overline{\mu}$ is complete.

Let $\lambda : \overline{\mathcal{M}} \to [0, \infty]$ be another measure which extends μ, and let $A \in \overline{\mathcal{M}}$. Then $A = E \cup F$ for some $E \in \mathcal{M}$ and F a subset of a measure zero set $N \in \mathcal{M}$. It follows that

$$\mu(E) = \lambda(E) \leq \lambda(A) \leq \lambda(E) + \lambda(F) \leq \lambda(E) + \lambda(N) = \mu(E) + \mu(N) = \mu(E).$$

Therefore $\lambda(A) = \mu(E) = \overline{\mu}(A)$, so $\lambda = \overline{\mu}$. This shows that $\overline{\mu}$ is the unique measure on $\overline{\mathcal{M}}$ which extends μ.

7. Set $\mu := \sum_{j=1}^n a_j \mu_j$, and note that $\mu(\varnothing) = 0$. If $\{E_i\}_{i \in \mathbb{N}}$ is a pairwise disjoint collection of members of $\mathcal{M}$, then

$$\mu(\cup_{i \in \mathbb{N}} E_i) = \sum_{j=1}^n a_j \mu_j(\cup_{i \in \mathbb{N}} E_i) = \sum_{j=1}^n a_j \sum_{i \in \mathbb{N}} \mu_j(E_i) = \sum_{i \in \mathbb{N}} \sum_{j=1}^n a_j \mu_j(E_i) = \sum_{i \in \mathbb{N}} \mu(E_i)$$

(note that we are allowed to change the order of summation because all the terms are non-negative; alternatively you can consider the convergent/divergent cases separately). This shows that μ is a measure on $(X, \mathcal{M})$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_n\}_{n \in \mathbb{N}}$ a countable collection of measurable sets. Then $(\cap_{k=n}^\infty E_k)_{n \in \mathbb{N}}$ is an increasing sequence of measurable sets with $\mu(\cap_{k=n}^\infty E_k) \leq \mu(E_n)$ for all $n \in \mathbb{N}$, so

$$\mu(\liminf_{n \in \mathbb{N}} E_n) = \mu(\cup_{n \in \mathbb{N}}(\cap_{k=n}^\infty E_k)) = \lim_{n \to \infty} \mu(\cap_{k=n}^\infty E_k) = \liminf_{n \to \infty} \mu(\cap_{k=n}^\infty E_k) \leq \liminf_{n \to \infty} \mu(E_n).$$

Moreover $(\cup_{k=n}^\infty E_k)_{n \in \mathbb{N}}$ is a decreasing sequence of measurable sets with $\mu(\cup_{k=n}^\infty E_k) \geq \mu(E_n)$, so

$$\mu(\limsup_{n \in \mathbb{N}} E_n) = \mu(\cap_{n \in \mathbb{N}}(\cup_{k=n}^\infty E_k)) = \lim_{n \to \infty} \mu(\cup_{k=n}^\infty E_k) = \limsup_{n \to \infty} \mu(\cup_{k=n}^\infty E_k) \geq \limsup_{n \to \infty} \mu(E_n)$$

provided that the first term of the sequence has finite measure, i.e. $\mu(\cup_{k=1}^\infty E_k) < \infty$.

9. Let $(X, \mathcal{M}, \mu)$ be a measure space and $E, F \in \mathcal{M}$. Then $E = (E \cap F) \sqcup (E \setminus F)$ and $(E \setminus F) \sqcup F = E \cup F$, so

$$\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \setminus F) + \mu(F) = \mu(E \cap F) + \mu(E \cup F).$$

10. Clearly $\mu_E(\varnothing) = \mu(\varnothing \cap E) = \mu(\varnothing) = 0$. If $\{E_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint collection of members of $\mathcal{M}$, then

$$\mu_E(\cup_{n \in \mathbb{N}} E_n) = \mu((\cup_{n \in \mathbb{N}} E_n) \cap E) = \mu(\cup_{n \in \mathbb{N}}(E_n \cap E)) = \sum_{n \in \mathbb{N}} \mu(E_n \cap E) = \sum_{n \in \mathbb{N}} \mu_E(E_n).$$

Therefore μ_E is a measure.

11. Suppose that μ is continuous from below. Let $\{E_n\}_{n \in \mathbb{N}}$ be a pairwise disjoint collection of measurable sets, and for each $n \in \mathbb{N}$ define $F_n := \cup_{i=1}^n E_n$. Since μ is continuous from below and finitely additive,

$$\mu(\cup_{i \in \mathbb{N}} E_i) = \mu(\cup_{n \in \mathbb{N}} F_n) = \lim_{n \to \infty} \mu(F_n) = \lim_{n \to \infty} \sum_{i=1}^n \mu(E_i) = \sum_{i=1}^\infty \mu(E_i).$$

This shows that μ is a measure. Conversely, if μ is a measure it is continuous from below by Theorem 1.8.

Now suppose that $\mu(X) < \infty$ and μ is continuous from above. If $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of measurable sets, then $(E_n^c)_{n \in \mathbb{N}}$ is a decreasing sequence of measurable sets, which implies that

$$\mu(\cup_{n \in \mathbb{N}} E_n) = \mu(X) - \mu((\cup_{n \in \mathbb{N}} E_n)^c) = \mu(X) - \mu(\cap_{n \in \mathbb{N}} E_n^c) = \mu(X) - \lim_{n \to \infty} \mu(E_n^c) = \lim_{n \to \infty} \mu(E_n).$$

This shows that μ is continuous from below, so it is a measure by the previous paragraph. Conversely, if μ is a (finite) measure it is continuous from above by Theorem 1.8.

14. Suppose for a contradiction that there exists $C \in (0,\infty)$ such that every measurable subset $F \subseteq E$ satisfies $\mu(F) \leq C$ or $\mu(F) = \infty$. Set $M := \sup\{\mu(F) \mid F \subseteq E \text{ is measurable and } \mu(F) < \infty\}$, and note that $0 \leq M \leq C$. For each $n \in \mathbb{N}$ there exists a measurable subset $E_n \subseteq E$ such that $M - n^{-1} \leq \mu(E_n) < \infty$. Set $F_n := \cup_{i=1}^n E_n$ for each $n \in \mathbb{N}$ and define $F := \cup_{n\in\mathbb{N}} F_n$. Note that $M - n^{-1} \leq \mu(E_n) \leq \mu(F)$ and also $\mu(F_n) \leq \sum_{i=1}^n \mu(E_n) < \infty$ for all $n \in \mathbb{N}$, so $M \leq \mu(F) = \lim_{n\to\infty} \mu(F_n) \leq M$. This shows that $\mu(F) = M$, so $\mu(E \setminus F) = \infty$. Since μ is semifinite, there exists a measurable subset $A \subseteq E \setminus F$ such that $0 < \mu(A) < \infty$. This contradicts the definition of M, because $A \cup F \subseteq E$ but $\mu(F) < \mu(A) + \mu(F) = \mu(A \cup F) < \infty$. Therefore, for any $C \in (0,\infty)$ there exists a measurable subset $F \subseteq E$ such that $C < \mu(F) < \infty$.

17. Let $A, B \subseteq X$ be disjoint μ^*-measurable sets, and let $E \subseteq X$. Then $(A \cup B) \cap A^c = B$ and hence

$$\mu^*(E \cap (A \cup B)) = \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c) = \mu^*(E \cap A) + \mu^*(E \cap B).$$

It follows by induction that $\sum_{k=1}^n \mu^*(E \cap A_k) = \mu^*(E \cap (\cup_{k=1}^n A_k)) \leq \mu^*(E \cap (\cup_{k=1}^\infty A_k))$ for all $n \in \mathbb{N}$. Therefore

$$\sum_{k=1}^\infty \mu^*(E \cap A_k) \leq \mu^*(E \cap (\cup_{k=1}^\infty A_k)),$$

which implies that $\mu^*(E \cap (\cup_{k=1}^\infty A_k)) = \sum_{k=1}^\infty \mu^*(E \cap A_k)$ because μ^* is subadditive.

18. (a) Let $E \subseteq X$ and $\varepsilon \in (0,\infty)$. If $\mu^*(E) = \infty$ then $X \in \mathcal{A} \subseteq \mathcal{A}_\sigma$, $E \subseteq X$ and $\mu^*(X) \leq \mu^*(E) + \varepsilon$. Otherwise

$$\mu^*(E) = \inf\left\{\sum_{n\in\mathbb{N}} \mu_0(A_n) \;\middle|\; A_n \in \mathcal{A} \text{ for all } n \in \mathbb{N} \text{ and } E \subseteq \cup_{n\in\mathbb{N}} A_n\right\},$$

so there exists a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $E \subseteq \cup_{n\in\mathbb{N}} A_n$ and $\sum_{n\in\mathbb{N}} \mu_0(A_n) \leq \mu^*(E) + \varepsilon$. If $A := \cup_{n\in\mathbb{N}} A_n$, then $A \in \mathcal{A}_\sigma$, $E \subseteq A$ and $\mu^*(A) \leq \sum_{n\in\mathbb{N}} \mu_0(A_n) \leq \mu^*(E) + \varepsilon$.

(b) Let $E \subseteq X$ such that $\mu^*(E) < \infty$. Suppose that E is μ^*-measurable. For each $n \in \mathbb{N}$ there exists $B_n \in \mathcal{A}_\sigma$ such that $E \subseteq B_n$ and $\mu^*(B_n) \leq \mu^*(E) + n^{-1}$. Define $B := \cap_{n\in\mathbb{N}} B_n$, so that $B \in \mathcal{A}_{\sigma\delta}$ and $E \subseteq B$. Since $\mu^*(E) < \infty$, it follows that $\mu^*(B \setminus E) \leq \mu^*(B_n \cap E^c) = \mu^*(B_n) - \mu^*(B_n \cap E) = \mu^*(B_n) - \mu^*(E) \leq n^{-1}$ for all $n \in \mathbb{N}$. This implies that $\mu^*(B \setminus E) = 0$.

Conversely, suppose there exists $B \in A_{\sigma\delta}$ such that $E \subseteq B$ and $\mu^*(B \setminus E) = 0$. If $F \subseteq X$, then

$$\mu^*(F \cap E) + \mu^*(F \cap E^c) \leq \mu^*(F \cap B) + \mu^*(F \cap B^c) + \mu^*(F \cap (B \setminus E)) \leq \mu^*(F \cap B) + \mu^*(F \cap B^c) = \mu^*(F)$$

because $B \in A_{\sigma\delta}$ is μ^*-measurable. Clearly $\mu^*(F) \leq \mu^*(F \cap E) + \mu^*(F \cap E^c)$, so E is μ^*-measurable.

19. If $E \subseteq X$ is μ^*-measurable, then $\mu^*(X) = \mu^*(X \cap E) + \mu^*(X \cap E^c) = \mu^*(E) + \mu^*(E^c)$ by definition, so $\mu_*(E) = \mu^*(E)$.

Conversely, let $E \subseteq X$ and suppose that $\mu^*(E) = \mu_*(E)$. By the previous exercise, for each $n \in \mathbb{N}$ there exist $A_n, B_n \in \mathcal{A}_\sigma$ such that $E \subseteq A_n$, $E^c \subseteq B_n$, $\mu^*(A_n) \leq \mu^*(E) + n^{-1}$ and $\mu^*(B_n) \leq \mu^*(E^c) + n^{-1}$. If $A := \cap_{n\in\mathbb{N}} A_n$, then

$$\begin{aligned}
\mu^*(A \setminus E) \leq \mu^*(A_n \cap B_n) &= \mu^*(A_n) - \mu^*(A_n \setminus B_n) \\
&= \mu^*(A_n) - \mu^*(B_n^c) \\
&= \mu^*(A_n) - (\mu^*(X) - \mu^*(B_n)) \\
&\leq \mu^*(E) + n^{-1} - \mu^*(X) + \mu^*(E^c) + n^{-1}
\end{aligned}$$

$$= \mu^*(E) - \mu_*(E) + 2n^{-1}$$
$$= 2n^{-1}$$

for all $n \in \mathbb{N}$, and hence $\mu^*(A \setminus E) = 0$. Moreover $E \subseteq A$ and $A \in \mathcal{A}_{\sigma\delta}$, so by the previous exercise E is μ^*-measurable.

20. (a) If $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}^*$ such that $E \subseteq \cup_{n\in\mathbb{N}} A_n$, then

$$\mu^*(E) \leq \mu^*(\cup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n) = \sum_{n\in\mathbb{N}} \overline{\mu}(A_n).$$

Therefore $\mu^*(E) \leq \mu^+(E)$, by definition. If there exists $A \in \mathcal{M}^*$ such that $E \subseteq A$ and $\mu^*(A) = \mu^*(E)$, then

$$\mu^+(E) \leq \overline{\mu}(A) + \sum_{n=2}^{\infty} \overline{\mu}(\varnothing) = \mu^*(A) = \mu^*(E),$$

so $\mu^*(E) = \mu^+(E)$. Conversely, suppose that $\mu^*(E) = \mu^+(E)$. By exercise 18 part (a), for each $n \in \mathbb{N}$ there exists $A_n \in \mathcal{M}^*_\sigma = \mathcal{M}^*$ such that $E \subseteq A_n$ and $\mu^+(A_n) \leq \mu^+(E) + n^{-1}$. Define $A := \cap_{n\in\mathbb{N}} A_n$, so that $A \in \mathcal{M}^*$ and $E \subseteq A$. Moreover $\mu^+(A) \leq \mu^+(A_n) \leq \mu^+(E) + n^{-1}$ for all $n \in \mathbb{N}$, so $\mu^+(A) \leq \mu^+(E)$. It follows that $\mu^*(A) = \mu^+(A) = \mu^+(E) = \mu^*(E)$, because $A \in \mathcal{M}^*$ and $E \subseteq A$.

(b) Let μ_0 be a premeasure on an algebra $\mathcal{A} \subseteq \mathcal{M}^*$ such that μ^* is induced by μ_0, and let $E \subseteq X$. By exercise 18 part (a), for each $n \in \mathbb{N}$ there exists $A_n \in \mathcal{A}_\sigma \subseteq \mathcal{M}^*$ such that $E \subseteq A_n$ and $\mu^*(A_n) \leq \mu^*(E) + n^{-1}$. Define $A := \cap_{n\in\mathbb{N}} A_n$, so that $A \in \mathcal{M}^*$ and $E \subseteq A$. Moreover $\mu^*(A) \leq \mu^*(A_n) \leq \mu^*(E) + n^{-1}$ for all $n \in \mathbb{N}$, so $\mu^*(A) \leq \mu^*(E)$. It follows that $\mu^*(A) = \mu^*(E)$, because $E \subseteq A$. By the previous exercise, this implies that $\mu^*(E) = \mu^+(E)$. Therefore $\mu^* = \mu^+$.

(c) Define $\mu^* : 2^X \to [0, \infty]$ by $\mu^*(\varnothing) = 0$, $\mu^*(\{0\}) = 2$, $\mu^*(\{1\}) = 2$ and $\mu^*(X) = 3$. Clearly $\mu^*(A) \leq \mu^*(B)$ for all $A \subseteq B \subseteq X$. Moreover μ^* is subadditive, because $\mu^*(X) < \mu^*(\{0\}) + \mu^*(\{1\})$. Therefore μ^* is an outer measure. Note that $\{0\}$ is not μ^*-measurable, because $\mu^*(X) = 3 < 2 + 2 = \mu^*(X \cap \{0\}) + \mu^*(X \cap \{0\}^c)$. This implies that $\mu^+(\{0\}) = \inf\{\sum_{n\in\mathbb{N}} \mu^*(A_n) \mid A_n \subseteq X \text{ for all } n \in \mathbb{N} \text{ and } A_n = X \text{ for some } n \in \mathbb{N}\} = \mu^*(X) = 3 \neq 2 = \mu^*(\{0\})$.

21. Let $E \subseteq X$ be locally μ^*-measurable, and let $A \subseteq X$. If $\mu^*(A) = \infty$, then clearly $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$. Otherwise, by exercise 18 part (a), for each $n \in \mathbb{N}$ there exists a μ^*-measurable set $A_n \subseteq X$ such that $A \subseteq A_n$ and $\mu^*(A_n) \leq \mu^*(A) + n^{-1}$. In particular, if $n \in \mathbb{N}$ then $\mu^*(A_n) < \infty$, and hence $A_n \cap E$ and $A_n \cap E^c = A_n \cap (A_n \cap E)^c$ are μ^*-measurable (because E is locally μ^*-measurable). It follows that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A_n \cap E) + \mu^*(A_n \cap E^c) = \overline{\mu}(A_n \cap E) + \overline{\mu}(A_n \cap E^c) = \overline{\mu}(A_n) = \mu^*(A_n) \leq \mu^*(A) + n^{-1}$$

for all $n \in \mathbb{N}$, so $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$. Clearly $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ in either case, so E is μ^*-measurable. This shows that $\overline{\mu}$ is saturated.

22. (a) Clearly $\overline{\mu}|_{\mathcal{M}} = \mu$, so by exercise 6 from section 1.3 it suffices to show that $\mathcal{M}^* = \overline{\mathcal{M}}$. To this end, let $A \in \overline{\mathcal{M}}$. Then $A = E \cup F$ for some $E \in \mathcal{M}$ and $F \subseteq X$ such that $F \subseteq B$ for some $B \in \mathcal{M}$ with $\mu(B) = 0$. Clearly $A \subseteq E \cup B \in \mathcal{M} = \mathcal{M}_{\sigma\delta}$ and $\mu^*((E \cup B) \setminus A) \leq \mu^*(B) = 0$. By exercise 18 part (c), it follows that $A \in \mathcal{M}^*$.

Conversely, let $A \in \mathcal{M}^*$. By exercise 18 part (c) there exists $B \in \mathcal{M}_{\sigma\delta} = \mathcal{M}$ such that $A \subseteq B$ and $\mu^*(B \setminus A) = 0$. Since $B \setminus A \in \mathcal{M}^*$, exercise 18 part (b) implies that $B \setminus A \subseteq E$ and $\mu^*(E \setminus (B \setminus A)) = 0$ for some $E \in \mathcal{M}$. Note that $B \setminus E \in \mathcal{M}$ and $A \cap E \in \mathcal{M}^*$. Moreover $\mu^*(A \cap E) \leq \mu^*(E \setminus (B \setminus A)) = 0$, so there exists $F \in \mathcal{M}$ such that $A \cap E \subseteq F$ and $\mu^*(F \setminus (A \cap E)) = 0$, by exercise 18 part (b). It follows that

$$\mu(F) = \mu^*(F) \leq \mu^*(F \setminus (A \cap E)) + \mu^*(A \cap E) = 0.$$

Since $A = (B \setminus E) \cup (A \cap E)$, it follows that $A \in \overline{\mathcal{M}}$. Therefore $\mathcal{M}^* = \overline{\mathcal{M}}$.

(b) Let $\widetilde{E} \in \widetilde{\overline{\mathcal{M}}}$, and $\widehat{\mu}$ be the completion of μ. Given $A \in \mathcal{M}^*$ with $\overline{\mu}(A) < \infty$, the converse of the previous exercise implies that $A \in \overline{\mathcal{M}}$ (using exercise 18 part (b) instead of part (c)). Hence $A = E \cup F$ for some $E \in \mathcal{M}$ and $F \subseteq X$ such that $F \subseteq B$ for some $B \in \mathcal{M}$ with $\mu(B) = 0$. In particular $\widehat{\mu}(A) = \mu(E) = \mu^*(E) \leq \mu^*(A) < \infty$. Therefore $\widetilde{E} \cap A \in \overline{\mathcal{M}}$, as $\widetilde{E}$ is locally $\widehat{\mu}$-measurable. Since $\mu^*(\widetilde{E} \cap A) < \infty$ the previous exercise implies that $\widetilde{E} \cap A \in \mathcal{M}^*$ (again using exercise 18 part (b) instead of part (c)). This shows that $\widetilde{E}$ is locally μ^*-measurable, so $\widetilde{E} \in \mathcal{M}^*$ by exercise 21.

Conversely, let $\widetilde{E} \in \mathcal{M}^*$ and $A \in \overline{\mathcal{M}}$ be such that $\widehat{\mu}(A) < \infty$. Then $A = E \cup F$ for some $E \in \mathcal{M}$ and $F \subseteq X$ such that $F \subseteq B$ for some $B \in \mathcal{M}$ with $\mu(B) = 0$. It follows that

$$\mu^*(\widetilde{E} \cap A) \leq \mu^*(A) \leq \mu^*(E) + \mu^*(F) = \mu(E) + 0 = \widehat{\mu}(A) < \infty.$$

This implies that $A \in \mathcal{M}^*$ (by the previous exercise using part (b) instead of part (c)), so $\widetilde{E} \cap A \in \mathcal{M}^*$ and hence $\widetilde{E} \cap A \in \overline{\mathcal{M}}$ (again by the previous exercise). This shows that $\widetilde{E}$ is locally $\widehat{\mu}$-measurable, so $\widetilde{E} \in \widetilde{\overline{\mathcal{M}}}$.

Therefore $\widetilde{\overline{\mathcal{M}}} = \mathcal{M}^*$. By exercise 6 from section 1.3, $\widetilde{\widehat{\mu}}$ and $\overline{\mu}$ agree on $\overline{\mathcal{M}}$. If $E \in \widetilde{\overline{\mathcal{M}}} \setminus \overline{\mathcal{M}}$, then $\widetilde{\widehat{\mu}}(E) = \infty$ by definition. Moreover, if $E \in \widetilde{\overline{\mathcal{M}}} = \mathcal{M}^*$ and $\overline{\mu}(E) < \infty$, then $E \in \overline{\mathcal{M}}$ by the previous exercise (using part (b) instead of part (c)). This implies that $\widetilde{\widehat{\mu}}$ and $\overline{\mu}$ also agree on $\widetilde{\overline{\mathcal{M}}} \setminus \overline{\mathcal{M}}$.

23. (a) Let $\mathcal{E} := \{(a, b] \cap \mathbb{Q} \mid a, b \in \overline{\mathbb{R}}\}$. Clearly $\varnothing = (0, 0] \cap \mathbb{Q} \in \mathcal{E}$. If $(a_1, b_1] \cap \mathbb{Q} \in \mathcal{E}$ and $(a_2, b_2] \cap \mathbb{Q} \in \mathcal{E}$ then their intersection is $(\max\{a_1, a_2\}, \min\{b_1, b_2\}] \cap \mathbb{Q} \in \mathcal{E}$. Moreover, the complement of $(a, b] \cap \mathbb{Q} \in \mathcal{E}$ is $((-\infty, a] \cap \mathbb{Q}) \cup ((b, \infty] \cap \mathbb{Q})$, which is a disjoint union of elements of $\mathcal{E}$ provided that $a \leq b$. If $a > b$ then the complement of $(a, b] \cap \mathbb{Q}$ is just $(-\infty, \infty] \cap \mathbb{Q} \in \mathcal{E}$. This shows that $\mathcal{E}$ is an elementary family of subsets of $\mathbb{Q}$, so the collection of finite disjoint unions of members of $\mathcal{E}$ is an algebra. If $(a_1, b_1] \cap \mathbb{Q} \in \mathcal{E}$ and $(a_2, b_2] \cap \mathbb{Q} \in \mathcal{E}$ are not disjoint, then their union is $(\min\{a_1, a_2\}, \max\{b_1, b_2\}] \cap \mathbb{Q} \in \mathcal{E}$. Therefore $\mathcal{A}$ is the collection of finite disjoint unions of members of $\mathcal{E}$, so $\mathcal{A}$ is an algebra.

(b) Let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Since $\mathbb{Q}$ is countable and $\mathcal{M}$ is closed under countable unions, it suffices to show that $\{x\} \in \mathcal{M}$ for all $x \in \mathbb{Q}$. Given $x \in \mathbb{Q}$ and $n \in \mathbb{N}$, it is clear that $(x - \frac{1}{n}, x] \cap \mathbb{Q} \in \mathcal{M}$. Therefore $\{x\} = \cap_{n\in\mathbb{N}}((x - \frac{1}{n}, x] \cap \mathbb{Q}) \in \mathcal{M}$, as required.

(c) By definition $\mu_0(\varnothing) = 0$. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of disjoint members of $\mathcal{A}$ whose union lies in $\mathcal{A}$. If $E_n = \varnothing$ for all $n \in \mathbb{N}$, then $\cup_{n\in\mathbb{N}}E_n = \varnothing$ and hence $\mu_0(\cup_{n\in\mathbb{N}}E_n) = 0 = \sum_{n\in\mathbb{N}} \mu_0(E_n)$. Otherwise $\cup_{n\in\mathbb{N}}E_n \neq \varnothing$, and $E_m \neq \varnothing$ for some $m \in \mathbb{N}$, so $\mu_0(\cup_{n\in\mathbb{N}}) = \infty$ and $\sum_{n\in\mathbb{N}} \mu_0(E_n) \geq \mu_0(E_m) = \infty$. Therefore μ_0 is a premeasure on $\mathcal{A}$. Define $\mu_1, \mu_2 : 2^{\mathbb{Q}} \to [0, \infty]$ by

$$\mu_1(E) = \begin{cases} \infty, & E \neq \varnothing \\ 0, & E = \varnothing \end{cases} \quad \text{and} \quad \mu_1(E) = \begin{cases} \infty, & E \text{ contains } 2^{-n}m \text{ for some } m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\mu_1|_{\mathcal{A}} = \mu_0 = \mu_2|_{\mathcal{A}}$, because every non-empty interval contains a rational of the form $2^{-n}m$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. For the same reason that μ_0 is a premeasure, μ_1 is a measure. A very similar argument implies that μ_2 is a measure. But $\mu_1 \neq \mu_2$ because $\mu_1(\{3^{-1}\}) = \infty$ whereas $\mu_2(\{3^{-1}\}) = 0$.

24. (a) Since $\mu(A) + \mu(A^c \cap B) = \mu(A \cup B) = \mu(B) + \mu(B^c \cap A)$, by symmetry it suffices to show that $\mu(A^c \cap B) = 0$. To this end, note that $E \subseteq A \cup B^c = (A^c \cap B)^c$ (if a member of E is not in B^c, it is in $B \cap E = A \cap E \subseteq A$). So

$$\mu(X) = \mu^*(X) = \mu^*(E) \leq \mu^*((A^c \cap B)^c) = \mu((A^c \cap B)^c) = \mu(X) - \mu(A^c \cap B),$$

and hence $\mu(A^c \cap B) = 0$ as required.

(b) It is clear that $\mathcal{M}_E$ is a σ-algebra on E and that $\nu(\varnothing) = 0$. Let $\{E_n\}_{n\in\mathbb{N}}$ be a pairwise disjoint subset of $\mathcal{M}_E$. For each $n \in \mathbb{N}$ there exists $A_n \in \mathcal{M}$ such that $E_n = A_n \cap E$. Set $A := \cup_{i\in\mathbb{N}} \cup_{j=i+1}^{\infty} (A_i \cap A_j)$ and define $B_n := A_n \setminus A$ for each $n \in \mathbb{N}$. It is easily checked that $\{B_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint subset of $\mathcal{M}$ and that $E_n = B_n \cap E$ for all $n \in \mathbb{N}$. Therefore

$$\nu(\cup_{n\in\mathbb{N}} E_n) = \nu(\cup_{n\in\mathbb{N}}(B_n \cap E)) = \nu((\cup_{n\in\mathbb{N}} B_n) \cap E) = \mu(\cup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n) = \sum_{n\in\mathbb{N}} \nu(E_n),$$

which shows that ν is a measure on $\mathcal{M}_E$.

25. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of compact intervals covering $\mathbb{R}$, and fix $n \in \mathbb{N}$. There exists a G_δ set $V_n \in \mathcal{M}_\mu$ and a null set $N_n \in \mathcal{M}_\mu$ such that $E \cap C_n = V_n \setminus N_n$. Let $(V_{nk})_{k\in\mathbb{N}}$ be a sequence of open sets such that $V_n = \cap_{k\in\mathbb{N}} V_{nk}$. For each $k \in \mathbb{N}$ define an open set $U_{nk} := V_{nk} \cup C_n^c$, and set $U := \cap_{n,k\in\mathbb{N}} U_{nk}$. Then U is a G_δ set and $U \setminus E \subseteq \cup_{n\in\mathbb{N}} N_n$. Indeed, if $x \in U \setminus E$ there exists $n \in \mathbb{N}$ such that $x \in C_n$, which implies that $x \in V_{nk}$ for all $k \in \mathbb{N}$ and hence $x \in V_n$ but $x \notin E \cap C_n = V_n \setminus N_n$. Moreover $E \subseteq U$, because $E \subseteq U_{nk}$ for all $n, k \in \mathbb{N}$. It follows that $E = U \setminus (U \setminus E)$, where $U \setminus E$ is a null set.

If $n \in \mathbb{N}$, then $E \cap C_n = H_n \cup N_n$ for some F_σ set $H_n \in \mathcal{M}_\mu$ and some null set $N_n \in \mathcal{M}_\mu$. Clearly $\cup_{n\in\mathbb{N}} H_n$ and $\cup_{n\in\mathbb{N}} N_n$ are respectively F_σ and null sets. Moreover $E = \cup_{n\in\mathbb{N}}(E \cap C_n) = (\cup_{n\in\mathbb{N}} H_n) \cup (\cup_{n\in\mathbb{N}} N_n)$.

26. Let $E \in \mathcal{M}_\mu$ and suppose that $\mu(E) < \infty$. Given $\varepsilon \in (0, \infty)$, there exists an open set $U \in \mathcal{M}_\mu$ such that $E \subseteq U$ and $\mu(U) < \mu(E) + \frac{\varepsilon}{2}$. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of disjoint open intervals such that $\cup_{n\in\mathbb{N}} U_n = U$. Then

$$\sum_{n\in\mathbb{N}} \mu(U_n) = \mu(U) < \infty,$$

so there exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \mu(U_n) < \frac{\varepsilon}{2}$. Define $A := \cup_{n=1}^{N} U_n$. It follows that

$$\mu(E \triangle A) \leq \mu(E \setminus A) + \mu(A \setminus E) \leq \mu(E \setminus U) + \mu(U \setminus A) + \mu(U \setminus E) = 0 + \sum_{n=N+1}^{\infty} \mu(U_n) + \mu(U) - \mu(E) < \varepsilon.$$

28. Let $a, b \in \mathbb{R}$. Since μ_F is continuous from above,

$$\mu_F([a,b]) = \mu_F(\cap_{n\in\mathbb{N}}(a - n^{-1}, b]) = \lim_{n\to\infty} \mu_F((a - n^{-1}, b]) = \lim_{n\to\infty} (F(b) - F(a - n^{-1})) = F(b) - F(a-).$$

It follows that $\mu_F(\{a\}) = \mu_F([a,a]) = F(a) - F(a-)$, in which case $\mu_F([a,b)) = \mu_F([a,b]) - \mu_F(\{b\}) = F(b-) - F(a-)$ and $\mu_F((a,b)) = \mu_F([a,b)) - \mu_F(\{a\}) = F(b-) - F(a)$.

29. (a) Suppose that $E \subseteq N$ but $m(E) > 0$. Define $R := \mathbb{Q} \cap [0,1)$, and for each $r \in R$ set $E_r := E + r$. Clearly each E_r is measurable with $m(E_r) = m(E)$, and $\cup_{r\in R} E_r \subseteq [0,2)$. Let $r, s \in R$ and suppose that E_r intersects E_s. Then there exists $t \in E_r \cap E_s$, so that $t - r, t - s \in E \subseteq N$. Since $t - r = (t - s) + (s - r)$ and $s - r \in \mathbb{Q}$, the definition of N implies that $t - s = t - r$. Therefore $r = s$, which shows that $\{E_r\}_{r\in R}$ is pairwise disjoint. Hence

$$\infty = \sum_{r\in R} m(E) = \sum_{r\in R} m(E_r) = m(\cup_{r\in R} E_r) \leq m([0,2)) = 2,$$

which is a contradiction. Therefore $m(E) = 0$.

(b) Suppose that $m(E) > 0$, but every subset of E is measurable. Since $E = \cup_{n\in\mathbb{Z}}(E \cap [n, n+1))$, there exists $n \in \mathbb{Z}$ such that $m(E \cap [n, n+1)) > 0$. Define $F := (E \cap [n, n+1)) - n$, so that $F \subseteq [0, 1)$, $m(F) > 0$ and every subset of F is measurable. Also define $R := \mathbb{Q} \cap [-1, 1]$, and for each $r \in R$ set $N_r := N + r$. It is clear that $[0, 1) \subseteq \cup_{r\in R} N_r$, and hence $F = \cup_{r\in R}(F \cap N_r)$. If $r \in R$ then $F \cap N_r \subseteq F$ and $(F \cap N_r) - r \subseteq N$, which implies that both subsets are measurable (by containment in F and translational invariance) and have measure zero (by the previous exercise and translational invariance). Therefore $m(F) \leq \sum_{r\in R} m(F \cap N_r) = \sum_{r\in R} 0 = 0$, which is a contradiction so not every subset of E is measurable.

30. Let $E \in \mathcal{L}$ with $m(E) > 0$, and suppose there exists $\alpha \in (0, 1)$ such that $m(E \cap I) \leq \alpha m(I)$ for all open intervals I. Without loss of generality $m(E) < \infty$ (since m is semifinite, we may replace E by a subset of finite positive measure). Define $\varepsilon := m(E)(1 - \alpha)$, so that $\varepsilon > 0$. Since $E \in \mathcal{L}$ and m is outer regular, there exists an open set $U \subseteq \mathbb{R}$ such that $E \subseteq U$ and $m(U) < m(E) + \varepsilon$. As U is open, there exists a pairwise disjoint collection $\{I_i\}_{i\in\mathbb{N}}$ of open intervals such that $U = \cup_{i\in\mathbb{N}} I_i$, and these intervals are bounded because $m(U) < \infty$. If if $i \in \mathbb{N}$ then

$$m(I_i) = m(I_i \setminus E) + m(E \cap I_i) \leq m(I_i \setminus E) + \alpha m(I_i)$$

and hence $(1 - \alpha)m(I_i) \leq m(I_i \setminus E)$. Since $\cup_{i\in\mathbb{N}}(I_i \setminus E) = U \setminus E$ has measure $m(U) - m(E)$, it follows that

$$(1-\alpha)m(U) = (1-\alpha)\sum_{i\in\mathbb{N}} m(I_i) = \sum_{i\in\mathbb{N}}(1-\alpha)m(I_i) \leq \sum_{i\in\mathbb{N}} m(I_i \setminus E) = m(\cup_{i\in\mathbb{N}}(I_i \setminus E)) < \varepsilon = (1-\alpha)m(E)$$

and hence $m(U) < m(E) \leq m(U)$, which is impossible. Thus, for each $\alpha \in (0, 1)$ there exists an open interval I such that $m(E \cap I) > \alpha m(I)$. The same clearly holds for $\alpha \in (-\infty, 0]$.

31. Let $E \in \mathcal{L}$ with $m(E) > 0$, and set $\alpha := \frac{3}{4}$. By the previous exercise there exists a open interval $I \subseteq \mathbb{R}$ with endpoints $a, b \in \mathbb{R}$ such that $m(E \cap I) > \alpha m(I)$. Suppose there exists $x \in (-\frac{1}{2}m(I), \frac{1}{2}m(I))$ such that $x \notin E - E$. Then $-x \notin E - E$, and clearly $0 \in E - E$, so we may assume $x > 0$. Moreover $E \subseteq (E^c + x) \cap (E^c - x)$. Note that $(a, a + 2x] \subseteq I$, because $2x < m(I) = b - a$. Define $n := \max\{k \in \mathbb{N} \mid a + 2kx < b\}$, so that

$$\begin{aligned}
m(E \cap (a, a+2nx]) &= \sum_{k=1}^{n}(m(E \cap (a+2(k-1)x, a+(2k-1)x]) + m(E \cap (a+(2k-1)x, a+2kx])) \\
&\leq \sum_{k=1}^{n}(m((E^c - x) \cap (a+2(k-1)x, a+(2k-1)x]) + m((E^c + x) \cap (a+(2k-1)x, a+2kx])) \\
&\leq \sum_{k=1}^{n}(m(E^c \cap (a+(2k-1)x, a+2kx] - x) + m(E^c \cap (a+2(k-1)x, a+(2k-1)x] + x)) \\
&\leq \sum_{k=1}^{n}(m(E^c \cap (a+(2k-1)x, a+2kx]) + m(E^c \cap (a+2(k-1)x, a+(2k-1)x])) \\
&\leq m(E^c \cap (a, a+2nx]).
\end{aligned}$$

This implies that $m(E \cap (a, a+2nx]) \leq \frac{1}{2}(m(E \cap (a, a+2nx]) + \frac{1}{2}m(E^c \cap (a, a+2nx])) = \frac{1}{2}m((a, a+2nx]) = nx$. Note that $4nx \geq m(I)$, since otherwise $k := 2n$ satisfies $a + 2kx < b$ (but $k > n$). It follows that

$$m(E \cap I) \leq m(E \cap (a, a+2nx]) + m((a+2nx, b)) \leq m(I) - nx \leq m(I) - \frac{m(I)}{4} = \frac{3m(I)}{4} = \alpha m(I).$$

This is a contradiction, so there does not exist $x \in (-\frac{1}{2}m(I), \frac{1}{2}m(I))$ such that $x \notin E - E$.

33. Choose a surjection $q : \mathbb{N} \to \mathbb{Q} \cap [0,1]$, and for each $n \in \mathbb{N}$ define $I_n \coloneqq (q(n) - 3^{-n}, q(n) + 3^{-n})$. For each $n \in \mathbb{N}$, define $D_n = I_n \setminus (\cup_{k=n+1}^{\infty} I_k)$. If $m, n \in \mathbb{N}$ and $m < n$, then $D_m \cap D_n = \varnothing$ because $D_n \subseteq I_n \subseteq \cup_{k=m+1}^{\infty} I_k$. Moreover

$$2 \cdot 3^{-n} = m(I_n) \leq m(D_n) + m(\cup_{k=n+1}^{\infty} I_k) \leq m(D_n) + \sum_{k=n+1}^{\infty} m(I_k) = m(D_n) + \sum_{k=n+1}^{\infty} 2 \cdot 3^{-k} = m(D_n) + 3^{-n},$$

and hence $m(D_n) \geq 3^{-n}$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ there exists a Borel set $A_n \subseteq D_n$ such that $0 < m(A_n) < 3^{-n}$, which can be found by intersecting D_n with an interval of the form $[3^{-n-1}k, 3^{-n-1}(k+1)]$ for some $k \in \mathbb{Z}$ (not all of the intersections can have zero measure). Define $A \coloneqq [0,1] \cap (\cup_{n \in \mathbb{N}} A_n)$, and let I be a subinterval of $[0,1]$ with midpoint c. There exists $n \in \mathbb{N}$ such that $4 \cdot 3^{-n} < m(I)$, and there are infinitely many rationals in $(c - 3^{-n}, c + 3^{-n})$, so there exists $k \in \mathbb{N}$ with $k \geq n$ and $q(k) \in (c - 3^{-n}, c + 3^{-n})$. It follows that $I_k \subseteq (c - 2 \cdot 3^{-n}, c + 2 \cdot 3^{n}) \subseteq I$. Therefore $A_k \subseteq A \cap I$ (since $A_k \subseteq I_k \subseteq [0,1]$), so $m(A \cap I) \geq m(A_k) > 0$. Moreover

$$m(A \cap I) = m((\cup_{n \in \mathbb{N}} A_n) \cap I) = m(A_k) + m((\cup_{n \in \mathbb{N} \setminus \{k\}} A_n) \cap I) < m(D_k) + m(I \setminus D_k) = m(I),$$

because $m(A_k) < m(D_k)$ and $\cup_{n \in \mathbb{N} \setminus \{k\}} A_n \subseteq \cup_{n \in \mathbb{N} \setminus \{k\}} D_n \subseteq D_k^c$.