1. Suppose f is measurable. Then $f^{-1}(\{-\infty\}) \in \mathcal{M}$ and $f^{-1}(\{\infty\}) \in \mathcal{M}$, because $\{-\infty\}$ and $\{\infty\}$ are Borel sets. If $B \subseteq \overline{R}$ is Borel then $f^{-1}(B) \in \mathcal{M}$, and hence $f^{-1}(B) \cap Y \in \mathcal{M}$ (since $\mathbb{R}$ is also Borel). Thus f is measurable on Y.

   Conversely, suppose that $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$ and f is measurable on Y. Let $B \subseteq \overline{\mathbb{R}}$ be Borel. Then $f^{-1}(B) \cap Y \in \mathcal{M}$, and $f^{-1}(B) = (f^{-1}(B) \cap Y) \cup (f^{-1}(B) \setminus Y)$. Clearly $f^{-1}(B) \setminus Y = f^{-1}(B \cap \{-\infty, \infty\})$, which is measurable because it is either $\varnothing$, $f^{-1}(\{-\infty\})$, $f^{-1}(\{-\infty\})$ or $f^{-1}(\{-\infty\}) \cup f^{-1}(\{-\infty\})$. This implies that $f^{-1}(B) \in \mathcal{M}$, so f is measurable.

3. If $f_n : X \to \overline{\mathbb{R}}$ for all $n \in \mathbb{N}$, then $g := \liminf_{n\to\infty} f_n$ and $h := \limsup_{n\to\infty} f_n$ are measurable. Therefore $f := h - g$ is measurable (setting $f(x) = 1$ whenever $g(x) = h(x) \in \{-\infty, \infty\}$), and hence

$$\{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\} = \{x \in X \mid \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) \notin \{-\infty, \infty\}\} = f^{-1}(\{0\}) = f^{-1}([-\infty, 0]) \in \mathcal{M}.$$

   If $f_n : X \to \mathbb{C}$ for all $n \in \mathbb{N}$ then $(f_n)_{n\in\mathbb{N}}$ converges at $x \in X$ iff $(\mathrm{Re}(f_n))_{n\in\mathbb{N}}$ and $(\mathrm{Im}(f_n))_{n\in\mathbb{N}}$ converge at x. So

$$\{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\} = \{x \in X \mid \lim_{n\to\infty} \mathrm{Re}(f_n(x)) \text{ exists}\} \cap \{x \in X \mid \lim_{n\to\infty} \mathrm{Im}(f_n(x)) \text{ exists}\} \in \mathcal{M}$$

4. If $a \in \mathbb{R}$ then there is a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{Q} \cap (a, \infty)$ converging to a, and $f^{-1}((a, \infty]) = \cup_{n\in\mathbb{N}} f^{-1}((a_n, \infty]) \in \mathcal{M}$. Since $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by such intervals $(a, \infty]$, it follows that f is measurable.

5. Suppose that f is measurable, and let E be a measurable set from the codomain of f. Then $f^{-1}(E) \in \mathcal{M}$, so $f^{-1}(E) \cap A, f^{-1}(E) \cap B \in \mathcal{M}$. Therefore f is measurable on A and on B.

   Conversely, suppose that f is measurable on A and on B. Again let E be a measurable set from the codomain of f. Then $f^{-1}(E) \cap A, f^{-1}(E) \cap B \in \mathcal{M}$, so $f^{-1}(E) = (f^{-1}(E) \cap A) \cup (f^{-1}(E) \cap B) \in \mathcal{M}$ and f is measurable.

6. For example set $X := \mathbb{R}$ and $\mathcal{M} := \mathcal{L}$. There exists a non-measurable set $A \subseteq X$, and for each $a \in A$ the set $\{a\}$ is measurable. Hence $\{\chi_{\{a\}}\}_{a\in A}$ is a family of measurable functions, but its supremum is χ_A, which is not measurable because $\chi_A^{-1}([1, \infty]) = A$.

8. Since f is measurable iff $-f$ is measurable, we may assume that f is increasing. Let $a \in \mathbb{R}$ and $x \in f^{-1}([a, \infty))$. If $y \in [x, \infty)$ then $f(y) \geq f(x) \geq a$ and hence $y \in f^{-1}([a, \infty))$. This shows that $f^{-1}([a, \infty))$ is an interval, so it is Borel measurable and hence f is Borel measurable.

9. (a) If $x, y \in [0, 1]$ and $x < y$, then $g(x) = f(x) + x \leq f(y) + x < f(y) + y = g(y)$ and hence g is injective. Since $g(0) = f(0) = 0$ and $g(1) = f(1) + 1 = 2$, the intermediate value theorem implies that g maps $[0, 1]$ onto $[0, 2]$. Let $x \in [0, 2]$ and $\varepsilon \in (0, \infty)$. Then there exists $x_0 \in [0, 1]$ such that $g(x_0) = x$. Define $x_1 := \max\{x_0 - \frac{1}{2}\varepsilon, 0\}$ and $x_2 := \min\{x_0 + \frac{1}{2}\varepsilon, 1\}$. Then $g(x_1) \leq x \leq g(x_2)$, and at least one of the inequalities is strict. Define $\delta := \min(\{x - g(x_1), g(x_2) - x\} \setminus \{0\})$. Given $y \in [0, 2]$ and $|y - x| < \delta$, it is straightforward to check that $g(x_1) \leq y \leq g(x_2)$. Indeed, if $x = g(x_1)$ then $g(x_1) = g(0) = 0$, and otherwise $g(x_1) \leq x - \delta$. Similarly $x_2 = 1$ or $x + \delta \leq g(x_2)$. Since g is increasing, it is clear that $x_1 \leq h(y) \leq x_2$. This implies that $x_0 - \frac{1}{2}\varepsilon \leq h(y) \leq x_0 + \frac{1}{2}\varepsilon$, so $|h(y) - h(x)| = |h(y) - x_0| < \varepsilon$. Therefore h is continuous on $[0, 2]$.

   (b) Note that $C = \{\sum_{n\in\mathbb{N}} 3^{-n} a_n \mid (a_n)_{n\in\mathbb{N}} \text{ is a sequence in } \{0, 2\}\}$, which implies that

$$g(C) = \left\{ f\left(\sum_{n\in\mathbb{N}} 3^{-n} a_n\right) + \sum_{n\in\mathbb{N}} 3^{-n} a_n \;\middle|\; (a_n)_{n\in\mathbb{N}} \text{ is a sequence in } \{0, 2\} \right\}$$

$$= \left\{ \sum_{n\in\mathbb{N}} 2^{-n}\frac{a_n}{2} + \sum_{n\in\mathbb{N}} 3^{-n}a_n \;\middle|\; (a_n)_{n\in\mathbb{N}} \text{ is a sequence in } \{0,2\} \right\}$$
$$= \left\{ \sum_{n\in\mathbb{N}} (2^{-n-1}+3^{-n})a_n \;\middle|\; (a_n)_{n\in\mathbb{N}} \text{ is a sequence in } \{0,2\} \right\}.$$

Set $C_0 := [0,2]$, and for each $n \in \mathbb{N}$ construct C_n from C_{n-1} by removing an open interval of length 3^{-n} from the middle of each interval comprising C_n. This works because C_{n-1} is the union of 2^{n-1} intervals of length $2^{1-n}+3^{1-n} > 3^{-n}$ (indeed, $2^0+3^0 = 2$ and $\frac{1}{2}(2^{1-n}+3^{1-n}-3^n) = 2^{-n}+3^{-n}$). Set $C' := \cap_{n\in\mathbb{N}} C_n$, so that

$$m(C') = \lim_{n\to\infty} m(C_n) = \lim_{n\to\infty} 2^n(2^{-n}+3^{-n}) = 1 + \lim_{n\to\infty} (\frac{2}{3})^n = 1.$$

Let $x \in g(C)$ and $N \in \mathbb{N}$. There exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $\{0,2\}$ such that $x = \sum_{n\in\mathbb{N}}(2^{-n-1}+3^{-n})a_n$. Clearly

$$0 \le x - \sum_{n=1}^{N}(2^{-n-1}+3^{-n})a_n \le \sum_{n=N+1}^{\infty} 2(2^{-n-1}+3^{-n}) = \frac{2^{-N-1}}{1-2^{-1}} + 2\frac{3^{-N-1}}{1-3^{-1}} = 2^{-N}+3^{-N}.$$

By induction on N it can be shown that $\sum_{n=1}^{N}(2^{-n-1}+3^{-n})a_n$ is the left endpoint of an interval from C_N, because the N^{th} term in the series is either 0 or $2^{-N}+2\cdot 3^{-N}$, the latter of which is the sum of length of the intervals in C_N and the length of the gaps between them. The above calculation therefore implies that $x \in C_N$. It follows that $x \in C'$, which shows that $g(C) \subseteq C'$.

Conversely, let $x \in C'$, so that $x \in C_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ define $a_n \in \{0,2\}$ depending on whether the interval x belongs to in C_n is the left or right child of its parent in C_{n-1}. Then x and $\sum_{n=1}^{N}(2^{-n-1}+3^{-n})a_n$ are from the same interval in C_N, for all $N \in \mathbb{N}$. In particular

$$\lim_{N\to\infty} \left| x - \sum_{n=1}^{N}(2^{-n-1}+3^{-n})a_n \right| \le \lim_{N\to\infty}(2^{-N}+3^{-N}) = 0,$$

which implies that $x = \sum_{n=1}^{\infty}(2^{-n-1}+3^{-n})a_n \in g(C)$. Therefore $C' \subseteq g(C)$, and hence $m(g(C)) = m(C') = 1$.

(c) Since $A \subseteq g(C)$, it is clear that $B \subseteq C$. Therefore $m^*(B) \le m^*(C) = 0$, so B is Lebesgue measurable. If B was Borel measurable, then $h^{-1}(B)$ would be as well, because h is continuous. However $h^{-1}(B) = A$, which is not Borel. Hence B is not Borel measurable.

(d) Set $F := \chi_B$ and $G := h$. Then F is Lebesgue measurable because $B \in \mathcal{L}$, and G is continuous by part (a). But

$$(F\circ G)^{-1}([1,\infty)) = \{x \in [0,2] \mid \chi_B(h(x)) \in [1,\infty)\} = \{x \in [0,2] \mid h(x) \in B\} = h^{-1}(B) = A \notin \mathcal{L},$$

so $F \circ G$ is not Lebesgue measurable.

11. If $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ then f_n is clearly Borel measurable on $[a_i, a_{i+1}]$, because $f_n|_{[a_i,a_{i+1}]}$ is the sum of products of Borel measurable functions. By an obvious generalisation of exercise 5, it follows that each f_n is Borel measurable. Let $(x,y) \in \mathbb{R}\times\mathbb{R}^k$ and $\varepsilon \in (0,\infty)$. Then there exists $\delta \in (0,\infty)$ such that $|f(x',y)-f(x,y)| < \varepsilon$ for all $x' \in (x-\delta, x+\delta)$. Moreover, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. Let $n \in \mathbb{N}$ with $n \ge N$, and choose $i \in \mathbb{Z}$ so that $x \in [a_i, a_{i+1}]$. Since $\frac{1}{n} < \delta$ it is clear that $a_i, a_{i+1} \in (x-\delta, x+\delta)$. Therefore

$$|f_n(x,y) - f(x,y)| = \left| \frac{f(a_{i+1},y)(x-a_i) - f(a_i,y)(x-a_{i+1}) - f(x,y)(a_{i+1}-a_i)}{a_{i+1}-a_i} \right|$$

$$
\begin{aligned}
&= \left| \frac{f(a_{i+1},y)(x-a_i) - f(a_i,y)(x-a_{i+1}) - f(x,y)(x-a_i) + f(x,y)(x-a_{i+1})}{a_{i+1}-a_i} \right| \\
&= \left| \frac{(f(a_{i+1},y)-f(x,y))(x-a_i) - (f(a_i,y)-f(x,y))(x-a_{i+1})}{a_{i+1}-a_i} \right| \\
&\le \left| \frac{(f(a_{i+1},y)-f(x,y))(x-a_i)}{a_{i+1}-a_i} \right| + \left| \frac{(f(a_i,y)-f(x,y))(x-a_{i+1})}{a_{i+1}-a_i} \right| \\
&= |f(a_{i+1},y)-f(x,y)| \cdot \frac{x-a_i}{a_{i+1}-a_i} + |f(a_i,y)-f(x,y)| \cdot \frac{a_{i+1}-x}{a_{i+1}-a_i} \\
&\le \varepsilon \cdot \frac{x-a_i}{a_{i+1}-a_i} + \varepsilon \cdot \frac{a_{i+1}-x}{a_{i+1}-a_i} \\
&= \varepsilon.
\end{aligned}
$$

This implies that $(f_n)_{n\in\mathbb{N}}$ converges to f pointwise, so f is Borel measurable. Clearly every function on $\mathbb{R}$ that is continuous in each variable is Borel measurable. Let $k \in \mathbb{N}$, and suppose that every function on $\mathbb{R}^k$ that is continuous in each variable is Borel measurable. Also let g be a function on $\mathbb{R}^{k+1}$ that is continuous in each variable. Then $g(x,\cdot)$ is a function on $\mathbb{R}^k$ that is continuous in each variable, and hence Borel measurable, for each $x \in \mathbb{R}$. From above, it follows that g is Borel measurable. By induction, for each $k \in \mathbb{N}$ every function on $\mathbb{R}^k$ that is continuous in each variable is Borel measurable.

13. Let $E \in \mathcal{M}$. By Fatou's lemma

$$\int_E f = \int f\chi_E = \int \liminf_{n\to\infty} f_n\chi_E \le \liminf_{n\to\infty} \int f_n\chi_E = \liminf_{n\to\infty} \int_E f_n$$

and similarly $\int_{E^c} f \le \liminf_{n\to\infty} \int_{E^c} f_n$. But $f\chi_E + f\chi_{E^c} = f$ and $f_n\chi_E + f_n\chi_{E^c} = f_n$ for all $n \in \mathbb{N}$, which implies that $\int_{E^c} f = \int f - \int_E f$ and (for sufficiently large $n \in \mathbb{N}$) $\int_{E^c} f_n = \int f_n - \int_E f_n$. Therefore

$$\int f - \int_E f = \int_{E^c} f \le \liminf_{n\to\infty} \int_{E^c} f_n = \liminf_{n\to\infty} \left(\int f_n - \int_E f_n \right) = \int f - \limsup_{n\to\infty} \int_E f_n,$$

so $\limsup_{n\to\infty} \int_E f_n \le \int_E f \le \liminf_{n\to\infty} \int_E f_n$ and hence $\int_E f = \lim_{n\to\infty} \int_E f_n$.

Define $F := (-\infty, 0)$ and for each $n \in \mathbb{N}$ set $F_n := F \cup [n, n+1)$. Then χ_F and each χ_{F_n} are in L^+, the sequence $(\chi_{F_n})_{n\in\mathbb{N}}$ converges to χ_F pointwise and $\int \chi_F = \infty = \lim_{n\to\infty} \int \chi_{F_n}$. However, $\int_{[0,\infty)} \chi_F = 0 \ne 1 = \lim_{n\to\infty} \int_{[0,\infty)} \chi_{F_n}$.

14. Clearly $\lambda(E) \ge 0$ for all $E \in \mathcal{M}$. Moreover, $\lambda(\varnothing) = \int_\varnothing f\,d\mu = \int f\chi_\varnothing\,d\mu = \int 0\,d\mu = 0$. If $\{E_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint subcollection of $\mathcal{M}$ then $(f\chi_{\cup_{n=1}^N E_n})_{N\in\mathbb{N}}$ is a sequence of measurable functions increasing to $f\chi_{\cup_{n\in\mathbb{N}} E_n}$, so

$$
\begin{aligned}
\lambda(\cup_{n\in\mathbb{N}} E_n) &= \int_{\cup_{n\in\mathbb{N}} E_n} f\,d\mu \\
&= \int f\chi_{\cup_{n\in\mathbb{N}} E_n}\,d\mu \\
&= \lim_{N\to\infty} \int f\chi_{\cup_{n=1}^N E_n}\,d\mu \\
&= \lim_{N\to\infty} \int f \sum_{n=1}^N \chi_{E_n}\,d\mu \\
&= \lim_{N\to\infty} \sum_{n=1}^N \int f\chi_{E_n}\,d\mu
\end{aligned}
$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \int_{E_n} f\,d\mu$$
$$= \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu$$
$$= \sum_{n=1}^{\infty} \lambda(E_n)$$

by the monotone convergence theorem. Therefore λ is a measure. Now let $g \in L^+$. If g is simple with standard representation $\sum_{n=1}^{N} a_n\chi_{E_n}$, then

$$\int g\,d\lambda = \sum_{n=1}^{N} a_n\lambda(E_n) = \sum_{n=1}^{N} a_n \int_{E_n} f\,d\mu = \sum_{n=1}^{N} a_n \int f\chi_{E_n}\,d\mu = \int \sum_{n=1}^{N} a_n f\chi_{E_n}\,d\mu = \int fg\,d\mu.$$

Otherwise, there exists an increasing sequence $(g_n)_{n\in\mathbb{N}}$ of simple functions in L^+ which converges pointwise to g, so that $(fg_n)_{n\in\mathbb{N}}$ increases pointwise to fg and hence

$$\int g\,d\lambda = \lim_{n\to\infty} \int g_n\,d\lambda = \lim_{n\to\infty} \int fg_n\,d\mu = \int fg\,d\mu,$$

by two applications of the monotone convergence theorem.

15. Since $\int f_1 < \infty$, the functions $\{f_n\}_{n\in\mathbb{N}}$ and f can be adjusted on a set of measure zero (namely $f_1^{-1}(\{\infty\})$) so that they map into $[0,\infty)$. This does not affect their integrals. Clearly $(f_1 - f_n)_{n\in\mathbb{N}}$ increases pointwise to $f_1 - f$. Moreover $f_1 - f_n \in L^+$ for all $n \in \mathbb{N}$. By the monotone convergence theorem $\int(f_1 - f) = \lim_{n\to\infty} \int(f_1 - f_n)$. Therefore

$$\int f = \int f + \int(f_1 - f) - \int(f_1 - f)$$
$$= \int f_1 - \lim_{n\to\infty} \int(f_1 - f_n)$$
$$= \lim_{n\to\infty} \left(\int f_1 - \int(f_1 - f_n)\right)$$
$$= \lim_{n\to\infty} \left(\int f_n + \int(f_1 - f_n) - \int(f_1 - f_n)\right)$$
$$= \lim_{n\to\infty} \int f_n,$$

since $\int(f_1 - f) \le \int f_1 < \infty$, and similarly $\int(f_1 - f_n) < \infty$ for all $n \in \mathbb{N}$.

16. For each $n \in \mathbb{N}$ define $E_n := \{x \in X \mid f(x) > n^{-1}\}$. Clearly $(f\chi_{E_n})_{n\in\mathbb{N}}$ increases pointwise to f, so by the monotone convergence theorem $(\int_{E_n} f)_{n\in\mathbb{N}}$ increases to $\int f$. In particular, given $\varepsilon \in (0,\infty)$ there exists $n \in \mathbb{N}$ such that $\int_{E_n} f > \int f - \varepsilon$. Since $\int f < \infty$ it is clear that $\mu(E_n) < \infty$.

17. Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence in L^+, and set $f := \lim_{n\to\infty} f$. Then $f \in L^+$, and by Fatou's lemma

$$\int f = \int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n.$$

Since $f_n \le f$ and hence $\int f_n \le \int f$ for all $n \in \mathbb{N}$, it is clear that $\limsup_{n\to\infty} \int f_n \le \int f$. Therefore

$$\limsup_{n\to\infty} \int f_n = \liminf_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n,$$

so $\int f = \lim_{n\to\infty} \int f_n$.

18. Let $g \in L^+ \cap L^1$, and $(f_n : X \to \overline{\mathbb{R}})_{n\in\mathbb{N}}$ be a sequence of measurable functions such that $f_n \geq -g$ for all $n \in \mathbb{N}$. Define $h := \liminf_{n\to\infty} f_n$. Clearly $h \geq -g$, so $h^-(x) = \max\{-h(x), 0\} \leq g(x)$ for all $x \in X$. It follows that $h^- \in L^1$ and $g - h^- \in L^+$. Similarly $f_n^- \in L^1$ and $g - f_n^-, f_n + g \in L^+$ for all $n \in \mathbb{N}$. Therefore, by Fatou's lemma

$$\begin{aligned}
\int h + \int g &= \int h^+ - \int h^- + \int g \\
&= \int h^+ + \int (g - h^-) \\
&= \int (h + g) \\
&= \int \liminf_{n\to\infty} (f_n + g) \\
&\leq \liminf_{n\to\infty} \int (f_n + g) \\
&= \liminf_{n\to\infty} \left(\int f_n^+ + \int (g - f_n^-) \right) \\
&= \liminf_{n\to\infty} \left(\int f_n^+ + \int g - \int f_n^- \right) \\
&= \liminf_{n\to\infty} \int f_n + \int g.
\end{aligned}$$

Since $\int g < \infty$, it follows that $\int \liminf_{n\to\infty} f_n = \int h \leq \liminf_{n\to\infty} \int f_n$.

Let $(f_n : X \to \overline{\mathbb{R}})_{n\in\mathbb{N}}$ be a sequence of nonpositive measurable functions. Define $h := \limsup_{n\to\infty} f_n$. Then $h \leq 0$ and $(-f_n)_{n\in\mathbb{N}}$ is a sequence in L^+, so by Fatou's lemma

$$-\int h = \int h^- - \int h^+ = \int -h = \int \liminf_{n\to\infty} -f_n \leq \liminf_{n\to\infty} \int -f_n = \liminf_{n\to\infty} \int f_n^- = \liminf_{n\to\infty} -\int f_n = -\limsup_{n\to\infty} \int f_n.$$

Therefore $\limsup_{n\to\infty} \int f_n \leq \int h = \int \limsup_{n\to\infty} f_n$.

19. (a) There exists $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| \leq 1$ for all $x \in X$ and $n \in \mathbb{N}$ with $n \geq N$. In particular

$$|f| = |f - f_N + f_N| \leq |f - f_N| + |f_N| \leq 1 + |f_N|,$$

and hence $\int |f| \, d\mu \leq \int 1 + |f_N| \, d\mu = \mu(X) + \int |f_N| \, d\mu < \infty$. This implies that $f \in L^1(\mu)$. Similarly $1 + |f| \in L^1(\mu)$. Since $|f_n| \leq 1 + |f|$ for all $n \in \mathbb{N}$ with $n \geq N$, the dominated convergence theorem implies that

$$\lim_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} \int f_{N+n} \, d\mu = \int \lim_{n\to\infty} f_{N+n} \, d\mu = \int f \, d\mu.$$

(b) For each $n \in \mathbb{N}$ define $f_n := 2^{-n} \chi_{[-2^n, 2^n]}$. Clearly $(f_n)_{n\in\mathbb{N}}$ converges uniformly to 0, but for each $n \in \mathbb{N}$

$$\int f_n \, d\mu = 2^{-n} \mu([-2^n, 2^n]) = 2 \neq 0 = \int 0 \, d\mu,$$

which implies that $f_n \in L^1(\mu)$ (where μ is the Lebesgue measure) and $\lim_{n\to\infty} \int f_n \, d\mu \neq \int 0 \, d\mu$.

20. It suffices to show that $\lim_{n\to\infty} \int \mathrm{Re}(f_n) = \int \mathrm{Re}(f)$ and $\lim_{n\to\infty} \int \mathrm{Im}(f_n) = \int \mathrm{Im}(f)$. Since $\lim_{n\to\infty} \mathrm{Re}(f_n) = \mathrm{Re}(f)$ and $\lim_{n\to\infty} \mathrm{Im}(f_n) = \mathrm{Im}(f)$ pointwise almost everywhere, while $|\mathrm{Re}(f_n)| \leq |f_n|$ and $|\mathrm{Im}(f_n)| \leq |f_n|$ for all $n \in \mathbb{N}$, we may assume without loss of generality that f and each f_n are real-valued. If $N, n \in \mathbb{N}$ such that $n \geq N$, then

$$\inf \left\{ \int g_m + \int f_m \right\}_{m=N}^{\infty} \leq \int g_n + \int f_n \leq \sup \left\{ \int g_m \right\}_{m=N}^{\infty} + \int f_n,$$

which implies that

$$\inf\left\{\int g_m + \int f_m\right\}_{m=N}^{\infty} \le \inf\left\{\sup\left\{\int g_m\right\}_{m=N}^{\infty} + \int f_n\right\}_{n=N}^{\infty} = \sup\left\{\int g_m\right\}_{m=N}^{\infty} + \inf\left\{\int f_m\right\}_{m=N}^{\infty}.$$

Therefore

$$\liminf_{n\to\infty}\left(\int g_n + \int f_n\right) \le \limsup_{n\to\infty}\int g_n + \liminf_{n\to\infty}\int f_n = \int g + \liminf_{n\to\infty}\int f_n.$$

Similarly

$$\liminf_{n\to\infty}\left(\int g_n - \int f_n\right) \le \int g + \liminf_{n\to\infty} - \int f_n = \int g - \limsup_{n\to\infty}\int f_n.$$

Since $g_n + f_n, g_n - f_n \in L^+$ for all $n \in \mathbb{N}$, Fatou's lemma implies that

$$\int g + \int f = \int (g+f) = \int \liminf_{n\to\infty}(g_n + f_n) \le \liminf_{n\to\infty}\int (g_n+f_n) = \liminf_{n\to\infty}\left(\int g_n + \int f_n\right) \le \int g + \liminf_{n\to\infty}\int f_n$$

and

$$\int g - \int f = \int (g-f) = \int \liminf_{n\to\infty}(g_n - f_n) \le \liminf_{n\to\infty}\int (g_n-f_n) = \liminf_{n\to\infty}\left(\int g_n - \int f_n\right) \le \int g - \limsup_{n\to\infty}\int f_n.$$

Since $\int g < \infty$, it follows that $\limsup_{n\to\infty}\int f_n \le \int f \le \liminf_{n\to\infty}\int f_n$, and hence $\int f = \lim_{n\to\infty}\int f_n$.

21. Suppose that $\lim_{n\to\infty}\int |f_n - f| = 0$. For every $\varepsilon \in (0,\infty)$, there exists $N \in \mathbb{N}$ such that

$$\left|\int |f_n| - \int |f|\right| = \left|\int (|f_n| - |f|)\right| \le \int \big||f_n| - |f|\big| \le \int |f_n - f| = \left|\int |f_n - f| - 0\right| < \varepsilon$$

for all $n \in \mathbb{N}$ with $n \ge N$ (by the reverse triangle inequality). Therefore $\lim_{n\to\infty}\int |f_n| = \int |f|$.

Conversely, suppose that $\lim_{n\to\infty}\int |f_n| = \int |f|$. For each $n \in \mathbb{N}$ it is clear that $|f_n|+|f| \in L^1$ and $|f_n - f| \le |f_n|+|f|$, so that $|f_n - f| \in L^1$. Moreover $(|f_n - f|)_{n\in\mathbb{N}}$ converges to $0 \in L^1$ pointwise almost everywhere. Also $(|f_n| + |f|)_{n\in\mathbb{N}}$ converges to $2|f| \in L^1$ pointwise almost everywhere, and

$$\lim_{n\to\infty}\int (|f_n| + |f|) = \lim_{n\to\infty}\int |f_n| + \int |f| = 2\int |f| = \int 2|f|.$$

Therefore, by the previous exercise, $\lim_{n\to\infty}\int |f_n - f| = \int 0 = 0$.

23. (a) Let $x \in [a,b]$, and suppose that $H(x) = h(x)$. Fix $\varepsilon \in (0,\infty)$. Since

$$\lim_{\delta\to 0^+}(\sup f([a,b]\cap[x-\delta,x+\delta]) - \inf f([a,b]\cap[x-\delta,x+\delta])) = H(x) - h(x) = 0,$$

there exists $\eta \in (0,\infty)$ such that $|\sup f([a,b]\cap[x-\delta,x+\delta]) - \inf f([a,b]\cap[x-\delta,x+\delta])| < \varepsilon$ for all $\delta \in (0,\eta)$, in particular for $\delta := \frac{\eta}{2}$. If $y \in [a,b]$ and $|x-y| < \delta$ then $y \in [a,b]\cap[x-\delta,x+\delta]$, so

$$f(x) - f(y) \le \sup f([a,b]\cap[x-\delta,x+\delta]) - \inf f([a,b]\cap[x-\delta,x+\delta]) < \varepsilon$$

and similarly $f(y) - f(x) < \varepsilon$, so $|f(x) - f(y)| < \varepsilon$. This shows that f is continuous at x.

Conversely, suppose that f is continuous at x. Let $\varepsilon \in (0,\infty)$. There exists $\eta \in (0,\infty)$ such that $|f(x) - f(y)| < \frac{\varepsilon}{3}$ for all $y \in [a,b]$ with $|x-y| < \eta$. Therefore $f(x) - \frac{\varepsilon}{3} < f(y) < f(x) + \frac{\varepsilon}{3}$ for all $y \in [a,b]\cap(x-\eta,x+\eta)$, so

$$\sup f([a,b]\cap[x-\delta,x+\delta]) - \inf f([a,b]\cap[x-\delta,x+\delta]) \le f(x) + \frac{\varepsilon}{3} - f(x) + \frac{\varepsilon}{3} < \varepsilon$$

for all $\delta \in (0,\eta)$. This shows that

$$H(x) - h(x) = \lim_{\delta\to 0^+}(\sup f([a,b]\cap[x-\delta,x+\delta]) - \inf f([a,b]\cap[x-\delta,x+\delta])) = 0.$$

(b) Choose a nested sequence $(P_n)_{n\in\mathbb{N}}$ of partitions of $[a,b]$ such that $(S_{P_n}f)_{n\in\mathbb{N}}$ converges to $\overline{I}_a^b(f)$. For each $n\in\mathbb{N}$ let E_n be the set of endpoints of the intervals comprising P_n, so that $m(E_n)=0$. Define $E := \cup_{n\in\mathbb{N}}E_n$, so that $m(E)=0$. Let $x\in[a,b]\setminus E$, and choose $\delta\in(0,\infty)$ such that

$$\sup f([a,b]\cap[x-\delta,x+\delta]) < H(x)+\frac{\varepsilon}{2}.$$

There exists $N\in\mathbb{N}$ such that $S_{P_n}f < \overline{I}_a^b(f)+\frac{\varepsilon\delta}{2}$ for all $n\in\mathbb{N}$ with $n\geq N$. Fix $n\in\mathbb{N}$ with $n\geq N$. There is an interval $[a',b']$ in P_n such that $x\in(a',b')$ (because $x\notin E_n$). If $[a',b']\subseteq[x-\delta,x+\delta]$ then

$$G_{P_n}(x)=\sup f([a',b'])\leq \sup f([a,b]\cap[x-\delta,x+\delta]) < H(x)+\frac{\varepsilon}{2} < H(x)+\varepsilon.$$

Otherwise $a' < x-\delta$ or $x+\delta < b'$, so that $[x-\delta,x]$ or $[x,x+\delta]$ is contained in (a',b'). Construct a new partition P_n' of $[a,b]$ from P_n by inserting x and one of $x-\delta$ or $x+\delta$ between a' and b'. In the former case

$$\begin{aligned}
S_{P_n'}f - S_{P_n}f &= \sup f([a',x-\delta])(x-\delta-a') + \sup f([x-\delta,x])\delta + \sup f([x,b'])(b'-x) - G_{P_n}(x)(b'-a') \\
&< G_{P_n}(x)(x-\delta-a') + \left(H(x)+\frac{\varepsilon}{2}\right)\delta + G_{P_n}(x)(b'-x) - G_{P_n}(x)(b'-a') \\
&= G_{P_n}(x)(x-\delta-a'+b'-x-b'+a') + \left(H(x)+\frac{\varepsilon}{2}\right)\delta \\
&= G_{P_n}(x)(-\delta) + \left(H(x)+\frac{\varepsilon}{2}\right)\delta \\
&= \left(H(x)-G_{P_n}(x)+\frac{\varepsilon}{2}\right)\delta,
\end{aligned}$$

which still holds for the latter case, by a similar calculation. It follows that

$$G_{P_n}(x) < \frac{1}{\delta}(S_{P_n}f - S_{P_n'}f) + H(x) + \frac{\varepsilon}{2} < \frac{1}{\delta}\left(\overline{I}_a^b(f)+\frac{\varepsilon\delta}{2} - S_{P_n'}f\right) + H(x)+\frac{\varepsilon}{2} \leq H(x)+\varepsilon.$$

Since $x\in(a',b')$, there exists $\eta\in(0,\infty)$ such that $[x-\eta,x+\eta]\subseteq(a',b')$. This implies that

$$H(x)=\inf_{\zeta\in(0,\infty)}\sup f([a,b]\cap[x-\zeta,x+\zeta]) \leq \sup f([a,b]\cap[x-\eta,x+\eta]) \leq \sup f([a',b']) = G_{P_n}(x).$$

Therefore $|G_{P_n}(x)-H(x)|<\varepsilon$, so $(G_{P_n}(x))_{n\in\mathbb{N}}$ converges to $H(x)$ and hence $(G_{P_n})_{n\in\mathbb{N}}$ converges to H pointwise almost everywhere. Since f is bounded and $m([a,b])<\infty$, the dominated convergence theorem implies that

$$\int_{[a,b]} H\,dm = \lim_{n\to\infty}\int G_{P_n}\,dm = \lim_{n\to\infty} S_{P_n}f = \overline{I}_a^b(f).$$

A similar argument implies that $(g_{P_n})_{n\in\mathbb{N}}$ converges to h pointwise almost everywhere, for all nested sequences $(P_n)_{n\in\mathbb{N}}$ of partitions of $[a,b]$ such that $(s_{P_n}f)_{n\in\mathbb{N}}$ converges to $\underline{I}_a^b(f)$. Therefore $\int_{[a,b]} h\,dm = \underline{I}_a^b(f)$.

25. (a) By the monotone convergence theorem and Theorem 2.28,

$$\int f = \lim_{n\to\infty}\int_{1/n}^1 x^{-1/2}\,dx = \lim_{n\to\infty} 2x^{1/2}\Big|_{1/n}^1 = \lim_{n\to\infty}(2-2n^{-1/2}) = 2. \tag{1}$$

Therefore $\int|g| = \sum_{n=1}^\infty 2^{-n}\int f(x-r_n)\,dx = \sum_{n=1}^\infty 2^{-n}\int f = 2$, by the monotone convergence theorem. It follows that $g\in L^1(m)$, and $g<\infty$ almost everywhere by Proposition 2.20.

(b) Let $E \subseteq \mathbb{R}$ be a null set and suppose that $h \in L^1(m)$ is equal to g on E^c. If $I \subseteq \mathbb{R}$ is an interval with at least two points, there exists $n \in \mathbb{N}$ such that r_n is an interior point of I. For each $k \in \mathbb{N}$ note that $(r_n, r_n + k^{-1}) \cap I$ has positive measure, so there exists $x_k \in ((r_n, r_n + k^{-1}) \cap I) \setminus E$. Clearly $\lim_{k\to\infty} x_k = r_n$, in which case $\lim_{k\to\infty} 2^{-n} f(x_k - r_n) = 2^{-n} \lim_{k\to\infty}(x_k - r_n)^{-1/2} = \infty$. But $2^{-n} f(x_k - r_n) \leq g(x_k) = h(x_k)$ for all $k \in \mathbb{N}$, which implies that h is unbounded on I. This shows that h is unbounded on every interval, so it is clearly everywhere discontinuous.

(c) By part (a) $g^2 < \infty$ almost everywhere. If $I \subseteq \mathbb{R}$ is an interval with at least two points, there exists $n \in \mathbb{N}$ such that r_n is an interior point of I. There exists $\delta \in (0, 1)$ such that $(r_n, r_n + \delta) \subseteq I$, and

$$\int_I g^2 \geq \int_{r_n}^{r_n+\delta} g^2 \geq \int_{r_n}^{r_n+\delta} 2^{-2n} f(x - r_n)^2 \, dx = 2^{-2n} \int_0^\delta f^2 = 2^{-2n} \int_0^\delta x^{-1} \, dx = \infty,$$

where the last step follows from an argument similar to (1) (and is even in the undergraduate calculus textbooks).

26. Let $x \in \mathbb{R}$ and $\varepsilon \in (0, \infty)$. For each $n \in \mathbb{N}$ define $f_n := |f|\chi_{[x-2^{-n},x+2^{-n}]}$, so that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^1(m)$ which is dominated by $|f| \in L^1(m)$. Moreover $(f_n)_{n\in\mathbb{N}}$ converges to 0 pointwise almost everywhere Therefore

$$\lim_{n\to\infty} \int f_n = \int 0 = 0,$$

by the dominated convergence theorem. Choose $n \in \mathbb{N}$ such that $\left|\int f_n\right| < \varepsilon$, and let $y \in (x - 2^{-n}, x + 2^{-n})$. Then

$$\begin{aligned}
|F(x) - F(y)| &= \left|\int_{-\infty}^x f(t)\,dt - \int_{-\infty}^y f(t)\,dt\right| \\
&= \left|\int f\chi_{(-\infty,x]} - \int f\chi_{(-\infty,y]}\right| \\
&= \left|\int f \cdot (\chi_{(-\infty,x]} - \chi_{(-\infty,y]})\right| \\
&\leq \int |f| \cdot |\chi_{(-\infty,x]} - \chi_{(-\infty,y]}| \\
&= \int |f| \cdot \chi_{[\min\{x,y\},\max\{x,y\}]} \\
&\leq \int |f| \cdot \chi_{[x-2^{-n},x+2^{-n}]} \\
&= \int f_n \\
&= \left|\int f_n\right| \\
&< \varepsilon.
\end{aligned}$$

This shows that then F is continuous at x, and hence F is continuous on $\mathbb{R}$.

28. (a) Fix $x \in [0, \infty)$ and define $f : [1, \infty) \to (0, 1]$ by $f(n) := (1 + \frac{x}{n})^{-n}$. Then

$$(\log \circ f)'(n) = -\log\left(1 + \frac{x}{n}\right) - \frac{n(-xn^{-2})}{1 + \frac{x}{n}} = \frac{x}{n + x} - \log\left(1 + \frac{x}{n}\right)$$

for all $n \in [1, \infty)$. Note that

$$\exp\left(\frac{x}{n+x}\right) = \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{x}{n+x}\right)^k \leq 1 + \sum_{k=1}^\infty \left(\frac{x}{n+x}\right)^k = 1 + \frac{\frac{x}{n+x}}{1 - \frac{x}{n+x}} = 1 + \frac{\frac{x}{n+x}}{\frac{n}{n+x}} = 1 + \frac{x}{n}$$

and hence $\frac{x}{n+x} \leq \log\left(1+\frac{x}{n}\right)$ for all $n \in [1,\infty)$, so $\log \circ f$ is a decreasing function. Therefore $f = \exp \circ \log \circ f$ is decreasing, so the sequence $(f_n)_{n\in\mathbb{N}}$ is decreasing as well, where each $f_n : [0,\infty) \to (0,1]$ is defined by $f_n(x) := (1+\frac{x}{n})^{-n}$. In particular $f_n \leq f_2$ for all $n \in \mathbb{N}$ with $n \geq 2$. By the monotone convergence theorem

$$\int_0^\infty f_2 = \lim_{n\to\infty} \int_0^\infty f_2 \chi_{[0,n]} = \lim_{n\to\infty} \int_0^n \left(1+\frac{x}{2}\right)^{-2} dx = \lim_{n\to\infty} \int_0^n \frac{d}{dx}\left(-2\left(1+\frac{x}{2}\right)^{-1}\right) dx,$$

so by the fundamental theorem of calculus

$$\int_0^\infty f_2 = \lim_{n\to\infty} \left(2(1+0)^{-1} - 2\left(1+\frac{n}{2}\right)^{-1}\right) = 2 - \lim_{n\to\infty} \frac{2}{1+\frac{n}{2}} = 2.$$

Therefore $f_2 \in L^1$, so (since $|\sin| \leq 1$) by the dominated convergence theorem

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx = \int_0^\infty \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx = \int_0^\infty 0\, dx = 0.$$

Indeed, if $x \in [0,\infty)$ then $\lim_{n\to\infty} \sin(\frac{x}{n}) = \sin(\lim_{n\to\infty} \frac{x}{n}) = \sin(0) = 0$, and

$$\begin{aligned}
\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} &= \exp\left(\lim_{n\to\infty}\left(-n\log\left(1+\frac{x}{n}\right)\right)\right) \\
&= \exp\left(-\lim_{n\to\infty}\left(\frac{\log\left(1+\frac{x}{n}\right)}{\frac{1}{n}}\right)\right) \\
&= \exp\left(-x\lim_{n\to\infty}\left(\frac{\log\left(1+\frac{x}{n}\right)-\log(1)}{\frac{x}{n}}\right)\right) \\
&= \exp\left(-x\log'(1)\right) \\
&= e^{-x}.
\end{aligned}$$

(b) If $x \in [0,1]$ and $n \in \mathbb{N}$ with $n \geq 1$ then $0 \leq (1+nx^2)(1+x^2)^{-n} \leq 1$ because

$$(1+x^2)^n = \sum_{k=0}^{n} \binom{n}{k}(x^2)^k = 1 + nx^2 + \sum_{k=2}^{n}\binom{n}{k}(x^2)^k \geq 1 + nx^2.$$

Moreover $\int_0^1 1\, dx = 1$. Hence, by the dominated convergence theorem

$$\lim_{n\to\infty} \int_0^1 (1+nx^2)(1+x^2)^{-n}\, dx = \int_0^1 \lim_{n\to\infty} (1+nx^2)(1+x^2)^{-n}\, dx = \int_0^1 0\, dx = 0.$$

Indeed, $(1+nx^2)(1+x^2)^{-n} \leq (1+nx^2)(1+nx^2+\binom{n}{2}x^4)^{-1}$ for all $x \in [0,1]$ and $n \in \mathbb{N}$ with $n \geq 2$, and

$$\lim_{n\to\infty} \frac{1+nx^2}{1+nx^2+\binom{n}{2}x^4} = \lim_{n\to\infty} \frac{1+nx^2}{1+nx^2+\frac{n(n-1)}{2}x^4} = \lim_{n\to\infty} \frac{n^{-2}+n^{-1}x^2}{n^{-2}+n^{-1}x^2+\frac{n-1}{2n}x^4} = \frac{0+0}{0+0+\frac{1}{2}x^4} = 0$$

for all $x \in (0,1]$ (and hence almost all $x \in [0,1]$).

(c) Define $f : [0,\infty) \to \mathbb{R}$ by $f(x) := \sin(x) - x$. Since $f(0) = 0$ and $f'(x) = \cos(x) - 1 \leq 0$ for all $x \in [0,\infty)$, this function is nonpositive and hence $\sin(x) \leq x$ for all $x \in [0,\infty)$. Moreover $\sin(x) \geq 0$ for all $x \in [0,1]$, so $-x \leq \sin(x)$ for all $x \in [0,\infty)$. By the monotone convergence theorem and the fundamental theorem of calculus,

$$\int_0^\infty (1+x^2)^{-1}\, dx = \lim_{n\to\infty} \int_0^\infty (1+x^2)^{-1}\chi_{[0,n]}(x)\, dx = \lim_{n\to\infty} \int_0^n \frac{d}{dx}\tan^{-1}(x)\, dx = \lim_{n\to\infty} (\tan^{-1}(n) - \tan^{-1}(0)) = \frac{\pi}{2}.$$

Since $|\sin(\frac{x}{n})/\frac{x}{n}| \leq 1$ for all $x \in [0, \infty)$ and $n \in \mathbb{N}$, the dominated convergence theorem implies that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty n \sin\left(\frac{x}{n}\right) (x(1+x^2))^{-1}\, dx &= \lim_{n\to\infty} \int_0^\infty \frac{\sin(\frac{x}{n})}{\frac{x}{n}} (1+x^2)^{-1}\, dx \\
&= \int_0^\infty \lim_{n\to\infty} \frac{\sin(\frac{x}{n})}{\frac{x}{n}} (1+x^2)^{-1}\, dx \\
&= \int_0^\infty \lim_{n\to\infty} \frac{\sin(\frac{x}{n}) - \sin(0)}{\frac{x}{n}} (1+x^2)^{-1}\, dx \\
&= \int_0^\infty \sin'(0)(1+x^2)^{-1}\, dx \\
&= \int_0^\infty \cos(0)(1+x^2)^{-1}\, dx \\
&= \int_0^\infty (1+x^2)^{-1}\, dx \\
&= \frac{\pi}{2}.
\end{aligned}$$

(d) Fix $n \in \mathbb{N}$. By the monotone convergence theorem and the fundamental theorem of calculus,

$$\int_a^\infty n(1+n^2x^2)^{-1}\, dx = \lim_{m\to\infty} \int_a^m \frac{d}{dx} \tan^{-1}(nx)\, dx = \lim_{m\to\infty} \tan^{-1}(nm) - \tan^{-1}(na) = \frac{\pi}{2} - \tan^{-1}(na).$$

Therefore

$$\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}\, dx = \begin{cases} 0, & a > 0 \\ \frac{\pi}{2}, & a = 0 \\ \pi, & a < 0. \end{cases}$$

This implies that there is no $f \in L^1$ such that $\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}\, dx = \int f$, by a similar argument to exercise 26. So the usual convergence theorem approach would not have helped to solve this exercise..

29. If $t \in (0, \infty)$, then (by the monotone convergence theorem and Theorem 2.28)

$$\int_0^\infty e^{-tx}\, dx = \lim_{k\to\infty} \int_0^k e^{-tx}\, dx = \lim_{k\to\infty} \frac{e^{-tx}}{-t}\bigg|_0^k = \lim_{k\to\infty} \left(-\frac{e^{-tk}}{t} + \frac{e^0}{t}\right) = \frac{1}{t}.$$

Given $n \in \mathbb{N}$, define $f_n : [0, \infty) \times [\frac{1}{2}, 2] \to [0, \infty)$ by $f_n(x, t) := x^n e^{-tx}$. We claim that $\int_0^\infty f_n(x, t)\, dx = n!\, t^{-(n+1)}$ for each $t \in [\frac{1}{2}, 2]$. By induction, we may assume that $\int_0^\infty f_{n-1}(x, t)\, dx = (n-1)!\, t^{-n}$. Note that

$$f_{n-1}(x, t) = x^{n-1} e^{-tx} \leq 3^{n-1}(n-1)!\, e^{x/3} e^{-tx} = 3^{n-1}(n-1)!\, e^{-(t-1/3)x} \leq 3^{n-1}(n-1)!\, e^{-(1/6)x}$$

for all $x \in [0, \infty)$, so $f_{n-1}(-, t) \in L^1([0, \infty))$. Moreover

$$\left|\frac{\partial}{\partial t} f_{n-1}(x, t)\right| = |-x^n e^{-tx}| = f_n(x, t) \leq 3^n n!\, e^{-(1/6)x}$$

for all $x \in [0, \infty)$, so by Theorem 2.27

$$\int_0^\infty f_n(x, t)\, dx = -\frac{\partial}{\partial t} \int_0^\infty f_{n-1}(x, t)\, dx = -\frac{\partial}{\partial t}(n-1)!\, t^{-n} = n(n-1)!\, t^{-(n+1)} = n!\, t^{-(n+1)},$$

as claimed. In particular $\int_0^\infty x^n e^{-x}\, dx = n!$.

If $t \in (0,\infty)$, then $\int_{-\infty}^{\infty} e^{-tx^2}\,dx = \sqrt{\pi/t}$. The easiest proof of this requires a theorem from later in the course, but if you are curious you can look up an alternative proof on Wikipedia. Given $n \in \mathbb{N}$, define $f_n : \mathbb{R} \times [\frac{1}{2}, 2] \to [0,\infty)$ by $f_n(x,t) := x^{2n}e^{-tx^2}$. We claim that $\int_{-\infty}^{\infty} f_n(x,t)\,dx = \frac{(2n)!}{4^n n!}\sqrt{\pi}t^{-(2n+1)/2}$ for each $t \in [\frac{1}{2}, 2]$. By induction, we may assume that $\int_{-\infty}^{\infty} f_{n-1}(x,t)\,dx = \frac{(2n-2)!}{4^{n-1}(n-1)!}\sqrt{\pi}t^{-(2n-1)/2}$. Note that

$$f_{n-1}(x,t) = x^{2n-2}e^{-tx^2} \leq 3^{n-1}(n-1)!\,e^{x^2/3}e^{-tx^2} = 3^{n-1}(n-1)!\,e^{-(t-1/3)x^2} \leq 3^{n-1}(n-1)!\,e^{-(1/6)x^2}$$

for all $x \in \mathbb{R}$, so $f_{n-1}(-,t) \in L^1(\mathbb{R})$. Moreover

$$\left|\frac{\partial}{\partial t} f_{n-1}(x,t)\right| = |-x^{2n}e^{-tx^2}| = f_n(x,t) \leq 3^n n!\,e^{-(1/6)x^2}$$

for all $x \in \mathbb{R}$, so by Theorem 2.27

$$\begin{aligned}\int_{-\infty}^{\infty} f_n(x,t)\,dx &= -\frac{\partial}{\partial t}\int_{-\infty}^{\infty} f_{n-1}(x,t)\,dx \\ &= -\frac{\partial}{\partial t}\frac{(2n-2)!}{4^{n-1}(n-1)!}\sqrt{\pi}t^{-(2n-1)/2} \\ &= \frac{(2n-2)!(2n-1)}{2\cdot 4^{n-1}(n-1)!}\sqrt{\pi}t^{-(2n+1)/2} \\ &= \frac{(2n)!}{4n\cdot 4^{n-1}(n-1)!}\sqrt{\pi}t^{-(2n+1)/2} \\ &= \frac{(2n)!}{4^n n!}\sqrt{\pi}t^{-(2n+1)/2},\end{aligned}$$

as claimed. In particular $\int_{-\infty}^{\infty} x^{2n}e^{-x^2}\,dx = \frac{(2n)!}{4^n n!}\sqrt{\pi}$.

30. For each $k \in \mathbb{N}$, we claim that $(1-k^{-1}x)^k \leq e^{-x}$ for all $x \in (0,k)$. If so, $x^n(1-k^{-1}x)^k\chi_{(0,k)}(x) \leq x^n e^{-x}$ for all $k \in \mathbb{N}$ and $x \in [0,\infty)$, and by the previous exercise we may apply the dominated convergence theorem to show that

$$\lim_{k\to\infty}\int_0^k x^n(1-k^{-1}x)^k\,dx = \int_0^{\infty}\lim_{k\to\infty} x^n(1-k^{-1}x)^k\chi_{(0,k)}\,dx = \int_0^k x^n e^{-x}\,dx = n!$$

(to prove that $\lim_{k\to\infty}(1-k^{-1}x)^k = e^{-x}$, take logarithms and apply l'Hôpital's rule). Now we prove the claim. It suffices to show that $k\log(1-k^{-1}x)+x \leq 0$ for all $x \in (0,k)$. This is certainly true for $x=0$. Moreover,

$$\frac{d}{dx}(k\log(1-k^{-1}x)+x) = \frac{k(-k^{-1})}{1-k^{-1}x}+1 = 1-\frac{1}{(1-k^{-1}x)} = \frac{-k^{-1}x}{1-k^{-1}x} < 0$$

for all $x \in (0,k)$. By the mean value theorem $k\log(1-k^{-1}x)+x = (k\log(1-k^{-1}x)+x)-(k\log(1-0)+0) < 0$ for all $x \in (0,k)$, which proves the claim.

33. There clearly exists a subsequence $(\int f_{n_k})_{k\in\mathbb{N}}$ of $(\int f_n)_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty}\int f_{n_k} = \liminf_{n\to\infty}\int f_n$. Moreover $(f_{n_k})_{k\in\mathbb{N}}$ converges to f in measure, because for every $\varepsilon \in (0,\infty)$

$$\lim_{k\to\infty}\mu(\{x \in X \mid |f_{n_k}(x)-f(x)| \geq \varepsilon\}) = \lim_{n\to\infty}\mu(\{x \in X \mid |f_n(x)-f(x)| \geq \varepsilon\}) = 0.$$

In particular $(f_{n_k})_{k\in\mathbb{N}}$ is Cauchy in measure, so it has a subsequence $(f_{n_{k_i}})_{i\in\mathbb{N}}$ which converges to a measurable function g pointwise almost everywhere. Clearly $(f_{n_{k_i}})_{i\in\mathbb{N}}$ also converges to g^+ pointwise almost everywhere. Moreover, $f = g^+$ almost everywhere because $(f_{n_{k_i}})_{i\in\mathbb{N}}$ converges in measure to both f and g^+ (thus $\mu(\{x \in X \mid |f(x)-g^+(x)| \geq \varepsilon\}) < \delta$ for all $\epsilon, \delta \in (0,\infty)$). Therefore, by Fatou's lemma

$$\int f = \int g^+ \leq \liminf_{i\to\infty}\int f_{n_{k_i}} = \lim_{i\to\infty}\int f_{n_{k_i}} = \lim_{k\to\infty}\int f_{n_k} = \liminf_{n\to\infty}\int f_n.$$

34. (a) It suffices to show that $\lim_{n\to\infty} \int \mathrm{Re}(f_n) = \int \mathrm{Re}(f)$ and $\lim_{n\to\infty} \int \mathrm{Im}(f_n) = \int \mathrm{Im}(f)$. Since

$$\{x \in X \mid |\mathrm{Re}(f_n)(x) - \mathrm{Re}(f)(x)| \geq \varepsilon\} \cup \{x \in X \mid |\mathrm{Im}(f_n)(x) - \mathrm{Im}(f)(x)| \geq \varepsilon\} \subseteq \{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}$$

for all $n \in \mathbb{N}$ and $\varepsilon \in (0, \infty)$, while $|\mathrm{Re}(f_n)| \leq |f_n|$ and $|\mathrm{Im}(f_n)| \leq |f_n|$ for all $n \in \mathbb{N}$, we may assume without loss of generality that f and each f_n are real-valued. Note that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, so it has a subsequence which converges pointwise almost everywhere to a measurable function which equals f almost everywhere. Therefore $f \in L^1$. Since $(g + f_n)_{n\in\mathbb{N}}$ and $(g - f_n)_{n\in\mathbb{N}}$ are sequences of non-negative measurable functions which converge in measure to $g + f$ and $g - f$ respectively, the previous exercise implies that

$$\int g + \int f = \int (g + f) \leq \liminf_{n\to\infty} \int (g + f_n) = \liminf_{n\to\infty} \left(\int g + \int f_n \right) = \int g + \liminf_{n\to\infty} \int f_n$$

and

$$\int g - \int f = \int (g - f) \leq \liminf_{n\to\infty} \int (g - f_n) = \liminf_{n\to\infty} \left(\int g - \int f_n \right) = \int g - \limsup_{n\to\infty} \int f_n$$

Since $\int g < \infty$, it follows that $\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n$, and hence $\int f = \lim_{n\to\infty} \int f_n$.

(b) Note that $(|f_n - f|)_{n\in\mathbb{N}}$ converges to 0 in measure, because

$$\{x \in X \mid \big||f_n(x) - f(x)| - 0\big| \geq \varepsilon\} = \{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}$$

for all $n \in \mathbb{N}$ and $\varepsilon \in (0, \infty)$. Moreover $|f_n - f| \leq |f_n| + |f| \leq 2g \in L^1$ for all $n \in \mathbb{N}$. Therefore, by part (a),

$$\lim_{n\to\infty} \|f_n - f\|_1 = \lim_{n\to\infty} \int |f_n - f| = \int 0 = 0.$$

This implies that $(f_n)_{n\in\mathbb{N}}$ converges to f in L^1.

35. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges to f in measure. For every $\varepsilon \in (0, \infty)$, $\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f| \geq \varepsilon\}) = 0$. In particular, for every $\varepsilon \in (0, \infty)$ there exists $N \in \mathbb{N}$ such that

$$0 - \varepsilon < \mu(\{x \in X \mid |f_n(x) - f| \geq \varepsilon\}) < 0 + \varepsilon = \varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq N$.

Conversely, suppose that, for every $\varepsilon \in (0, \infty)$, there exists $N \in \mathbb{N}$ such that $\mu(\{x \in X \mid |f_n(x) - f| \geq \varepsilon\}) < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq N$. Let $\varepsilon \in (0, \infty)$ and $\delta \in (0, \infty)$. Define $\eta := \min\{\varepsilon, \delta\}$. There exists $N \in \mathbb{N}$ such that $\mu(\{x \in X \mid |f_n(x) - f| \geq \eta\}) < \eta$ for all $n \in \mathbb{N}$ with $n \geq N$. Therefore

$$\mu(\{x \in X \mid |f_n(x) - f| \geq \varepsilon\}) \leq \mu(\{x \in X \mid |f_n(x) - f| \geq \eta\}) < \eta \leq \delta$$

for all $n \in \mathbb{N}$ with $n \geq N$, which implies that $\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f| \geq \varepsilon\}) = 0$. This shows that $(f_n)_{n\in\mathbb{N}}$ converges to f in measure.

37. (a) Let $x \in X$ be a point where $(f_n)_{n\in\mathbb{N}}$ converges to f. Then

$$\lim_{n\to\infty} \phi(f_n(x)) = \phi(\lim_{n\to\infty} f_n(x)) = \phi(f(x)),$$

so $(\phi \circ f_n)_{n\in\mathbb{N}}$ converges to $\phi \circ f$ on the same set that $(f_n)_{n\in\mathbb{N}}$ converges to f.

(b) Suppose that $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly, and let $\varepsilon \in (0,\infty)$. Since ϕ is uniformly continuous, there exists $\delta \in (0,\infty)$ such that $|\phi(w) - \phi(z)| < \varepsilon$ for all $w, z \in \mathbb{C}$ with $|w - z| < \delta$. Moreover, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \delta$ for all $x \in X$ and $n \in \mathbb{N}$ with $n \geq N$. Therefore $|\phi(f_n(x)) - \phi(f(x))| < \varepsilon$ for all $x \in X$ and $n \in \mathbb{N}$ with $n \geq N$. This shows that $(\phi \circ f_n)_{n\in\mathbb{N}}$ converges to $\phi \circ f$ uniformly.

Now suppose that $(f_n)_{n\in\mathbb{N}}$ converges to f almost uniformly. For every $\varepsilon \in (0,\infty)$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on E^c, and hence $(\phi \circ f_n)_{n\in\mathbb{N}}$ converges to $\phi \circ f$ uniformly on E^c by the previous argument. This shows that $(\phi \circ f_n)_{n\in\mathbb{N}}$ converges to $\phi \circ f$ almost uniformly.

Finally, suppose that $(f_n)_{n\in\mathbb{N}}$ converges to f in measure. Let $\varepsilon \in (0,\infty)$. There exists $\delta \in (0,\infty)$ such that $|\phi(w) - \phi(z)| < \varepsilon$ for all $w, z \in \mathbb{C}$ with $|w - z| < \delta$. Moreover

$$\lim_{n\to\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \delta\}) = 0.$$

Clearly $\{x \in X \mid |\phi(f_n(x)) - \phi(f(x))| \geq \varepsilon\} \subseteq \{x \in X \mid |f_n(x) - f(x)| \geq \delta\}$ for all $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} \mu(\{x \in X \mid |\phi(f_n(x)) - \phi(f(x))| \geq \varepsilon\}) = 0$$

and hence $(\phi \circ f_n)_{n\in\mathbb{N}}$ converges to $\phi \circ f$ in measure.

(c) For each $n \in \mathbb{N}$ define a measurable function $f_n : \mathbb{R} \to \mathbb{C}$ by $f_n(x) := 2^{-n}$. Also define $\phi : \mathbb{C} \to \mathbb{C}$ by

$$\phi(z) := \begin{cases} 1, & \mathrm{Re}(z) > 0 \\ -1, & \mathrm{Re}(z) \leq 0. \end{cases}$$

For all $x \in \mathbb{R}$ $(f_n(x))_{n\in\mathbb{N}}$ converges to 0, but $(\phi(f_n(x)))_{n\in\mathbb{N}} = (1)_{n\in\mathbb{N}}$ does not converge to to $\phi(0) = -1$.

Now define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) := x$, and for each $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{C}$ by $f_n(x) := x + 2^{-n}$. Clearly $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions converging uniformly, almost uniformly and in measure to the measurable function f. Define $\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(z) := z^2$. Let $E \subseteq \mathbb{R}$ and suppose that $(\phi \circ f_n)_{n\in\mathbb{N}}$ converges to $\phi \circ f$ uniformly on E. Then there exists $N \in \mathbb{N}$ such that $|\phi(f_n(x)) - \phi(f(x))| < 1$ for all $x \in E$ and $n \in \mathbb{N}$ with $n \geq N$ Since

$$|\phi(f_N(x)) - \phi(f(x))| = |(x + 2^{-N})^2 - x^2| = |2^{1-N}x + 2^{-2N}|$$

for all $x \in E$, it follows that $E \subseteq (-2^{N-1} - 2^{-N-1}, 2^{N-1} - 2^{-N-1})$. In particular $\mu(E^c) = \infty$, so $(\phi \circ f_n)_{n\in\mathbb{N}}$ does not converge to $\phi \circ f$ uniformly or almost uniformly. If $\varepsilon \in (0,\infty)$ and $n \in \mathbb{N}$, then

$$[2^{n-1}\varepsilon, \infty) \subseteq \{x \in \mathbb{R} \mid |\phi(f_n(x)) - \phi(f(x))| \geq \varepsilon\}$$

because $|\phi(f_n(x)) - \phi(f(x))| = 2^{1-n}x + 2^{-2n} \geq \varepsilon + 2^{-2n}$ for all $x \in [2^{n-1}\varepsilon, \infty)$. Therefore

$$\lim_{n\to\infty} \mu(\{x \in \mathbb{R} \mid |\phi(f_n(x)) - \phi(f(x))| \geq \varepsilon\}) = \infty,$$

so $(\phi \circ f_n)_{n\in\mathbb{N}}$ does not converge to $\phi \circ f$ in measure.

39. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions which converges to f almost uniformly. For each $n \in \mathbb{N}$ there exists $E_n \in \mathcal{M}$ such that $\mu(E_n) < 2^{-n}$ and $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly (hence pointwise) on E_n^c. Define $E := \cap_{n\in\mathbb{N}} \cup_{k=n}^\infty E_k$. Then $\mu(E) = \lim_{n\to\infty} \mu(\cup_{k=n}^\infty E_k) = 0$, since $\mu(\cup_{k=n}^\infty E_k) \leq 2^{1-n}$ for all $n \in \mathbb{N}$. Moreover, if $x \in E^c$ there exists $n \in \mathbb{N}$ such that $x \in E_n^c$ and hence $\lim_{k\to\infty} f_k(x) = f(x)$. Therefore $(f_n)_{n\in\mathbb{N}}$ converges to f pointwise almost everywhere.

Let $\varepsilon \in (0,\infty)$ and take $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on E^c. There exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in E^c$ and $n \in \mathbb{N}$ with $n \geq N$. It follows that $\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\} \subseteq E$, and hence $\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) \leq \mu(E) < \varepsilon$. By exercise 35, $(f_n)_{n\in\mathbb{N}}$ converges to f in measure.

40. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued measurable functions that converge pointwise to some $f : X \to \mathbb{C}$ on a set $A \subseteq X$ with $\mu(A^c) = 0$. Suppose there exists $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then $|f(x)| \leq g(x)$ for all $x \in A$. Fix $k \in \mathbb{N}$, and for each $n \in \mathbb{N}$ define $E_{k,n} := \cup_{m=n}^{\infty}\{x \in A \mid |f_m(x) - f(x)| \geq 2k^{-1}\}$. If $x \in E_{k,1}$ there exists $m \in \mathbb{N}$ such that $|f_m(x) - f(x)| \geq 2k^{-1}$ and hence $2g(x) \geq |f_m(x)| + |f(x)| \geq 2k^{-1}$. Therefore $k^{-1}\chi_{E_{k,1}} \leq g$, so $k^{-1}\chi_{E_{k,1}} \in L^1$ and hence $\mu(E_{k,1}) < \infty$. Since $\cap_{n\in\mathbb{N}} E_{k,n} \subseteq A \cap A^c = \varnothing$, it follows that $\lim_{n\to\infty} \mu(E_{n,k}) = 0$. Now let $\varepsilon \in (0, \infty)$ and for each $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ so that $\mu(E_{n_k,k}) < 2^{-k}\varepsilon$. Define $E := (\cup_{k\in\mathbb{N}} E_{n_k,k}) \cup A^c$. Then $\mu(E) < \varepsilon$, and for each $\delta \in (0, \infty)$ there exists $k \in \mathbb{N}$ such that $2k^{-1} < \delta$, whence $|f_m(x) - f(x)| < 2k^{-1} < \delta$ for all $x \in E^c$ and $m \in \mathbb{N}$ with $m \geq n_k$. This implies that $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on E^c.

42. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges to f in measure. Given $\varepsilon \in (0, \infty)$, there exists $N \in \mathbb{N}$ such that

$$\mu(\{x \in \mathbb{N} \mid \varepsilon \leq |f_n(x) - f(x)|\}) < 1$$

for all $n \in \mathbb{N}$ with $n \geq N$. This implies that $\{x \in \mathbb{N} \mid \varepsilon \leq |f_n(x) - f(x)|\} = \varnothing$, and hence $\|f_n - f\|_u \leq \varepsilon$, for all $n \in \mathbb{N}$ with $n \geq N$. Therefore $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Now suppose that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. If $\varepsilon \in (0, \infty)$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x, n \in \mathbb{N}$ with $n \geq N$. In particular $\mu(\{x \in \mathbb{N} \mid \varepsilon \leq |f_n(x) - f(x)|\}) = 0$ for all $n \in \mathbb{N}$ with $n \geq N$, which implies that $(f_n)_{n\in\mathbb{N}}$ converges to f in measure.

44. For each $n \in \mathbb{N}$ define $E_n := f^{-1}(B_n(0)) = \{x \in [a, b] \mid |f(x)| < n\}$. Then $\lim_{n\to\infty} \mu(E_n) = \mu(\cup_{n\in\mathbb{N}} E_n) = \mu([a, b])$, so there exists $m \in \mathbb{N}$ such that $\mu([a, b]) - \mu(E_m) < \frac{\varepsilon}{3}$. Define $g : \mathbb{R} \to \mathbb{C}$ by

$$g(x) := \begin{cases} f(x), & x \in E_m \\ 0, & x \in E_m^c. \end{cases}$$

Then $|g| \leq m\chi_{E_m} \leq m\chi_{[a,b]}$, so $g \in L^1(\mu)$. Hence for each $n \in \mathbb{N}$ there exists a compactly supported continuous function $g_n : \mathbb{R} \to \mathbb{C}$ such that $\|g_n - g\|_1 < n^{-1}$. Clearly $(g_n)_{n\in\mathbb{N}}$ converges to g in measure, so there exists a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ which converges to g pointwise almost everywhere. After restricting these functions to $[a, b]$, Egoroff's theorem implies that there exists $F \subseteq [a, b]$ such that $\mu(F) < \frac{\varepsilon}{3}$ and $(g_{n_k})_{k\in\mathbb{N}}$ converges to g uniformly on $[a, b] \setminus F$. By inner regularity there exists a compact set $E \subseteq E_m \setminus F$ such that $\mu(E) > \mu(E_m \setminus F) - \frac{\varepsilon}{3}$ and hence

$$\mu([a, b] \setminus E) = \mu([a, b]) - \mu(E) < \mu([a, b]) - \mu(E_m \setminus F) + \frac{\varepsilon}{3} \leq \mu([a, b]) + \mu(F) - \mu(E_m) + \frac{\varepsilon}{3} < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Moreover $(g_{n_k})_{k\in\mathbb{N}}$ converges to g uniformly on E, so $f|_E = g|_E$ is continuous.

46. Fix $y \in Y$. Then $\chi_D(x, y) = \chi_{\{y\}}(x)$ for all $x \in X$, so $\int \chi_D(x, y)\, d\mu(x) = \mu(\{y\}) = 0$. This implies that

$$\iint \chi_D(x, y)\, d\mu(x)\, d\nu(y) = \int 0\, d\nu(y) = 0.$$

Now fix $x \in X$. Clearly $\chi_D(x, y) = \chi_{\{x\}}(y)$ for all $y \in Y$, so $\int \chi_D(x, y)\, d\nu(y) = \nu(\{x\}) = 1$. It follows that

$$\iint \chi_D(x, y)\, d\nu(y)\, d\mu(x) = \int 1\, d\mu(x) = \mu(X) = 1.$$

By definition $\int \chi_D\, d(\mu \times \nu) = (\mu \times \nu)(D)$, and hence

$$\int \chi_D\, d(\mu \times \nu) = \inf \left\{ \sum_{n=1}^{\infty} (\mu \times \nu)(E_n) \;\middle|\; (E_n)_{n=1}^{\infty} \text{ is a sequence of finite disjoint unions of rectangles covering } D \right\}$$

$$= \inf\left\{\sum_{n=1}^{\infty}\mu(A_n)\nu(B_n)\;\middle|\;(A_n\times B_n)_{n=1}^{\infty}\text{ is a sequence of rectangles covering }D\right\}$$

If $(A_n \times B_n)_{n=1}^{\infty}$ is a sequence of rectangles covering D, then $(A_n \cap B_n)_{n=1}^{\infty}$ covers X. Clearly this implies that $\mu^*(A_n \cap B_n) > 0$ for some $n \in \mathbb{N}$. In particular $\mu(A_n) > 0$ and $\nu(B_n) = \infty$, because the Lebesgue outer measure of a finite set is 0. Therefore $\sum_{n=1}^{\infty}\mu(A_n)\nu(B_n) = \infty$, so $\int \chi_D\, d(\mu\times\nu) = \inf\{\infty\} = \infty$.

48. Clearly $\int |f|\, d(\mu \times \nu) = (\mu \times \nu)(\cup_{n=1}^{\infty}\{(n,n),(n+1,n)\})$. If $(A_n \times B_n)_{n=1}^{\infty}$ is a sequence of rectangles covering $\cup_{n=1}^{\infty}\{(n,n),(n+1,n)\}$, then $(A_n \cap B_n)_{n=1}^{\infty}$ covers $\mathbb{N}$ and hence $\sum_{n=1}^{\infty}\mu(A_n\cap B_n) = \infty$. This implies that

$$\sum_{n=1}^{\infty}\mu(A_n)\nu(B_n) = \sum_{n=1}^{\infty}\mu(A_n)\mu(B_n) \geq \sum_{n=1}^{\infty}\mu(A_n\cap B_n)^2 \geq \sum_{n=1}^{\infty}\mu(A_n\cap B_n) = \infty,$$

since $\mu(A_n \cap B_n) \in \{0\} \cup [1,\infty]$ for all $n \in \mathbb{N}$. Therefore $\int |f|\, d(\mu\times\nu) = \inf\{\infty\} = \infty$. Fix $n \in Y$. Then $f(m,n) = \chi_{\{n\}}(m) - \chi_{\{n+1\}}(m)$ for all $m \in X$, and hence $\int f(m,n)\, d\mu(m) = \mu(\{n\}) - \mu(\{n+1\}) = 0$. This implies that

$$\iint f(m,n)\, d\mu(m)\, d\nu(n) = \int 0\, d\nu(n) = 0.$$

Now fix $m \in X\backslash\{1\}$. Then $f(m,n) = \chi_{\{m\}}(n) - \chi_{\{m-1\}}(n)$ for all $n \in Y$, so $\int f(m,n)\, d\nu(n) = \nu(\{m\}) - \nu(\{m-1\}) = 0$. Moreover, $\int f(1,n)\, d\nu(n) = \int \chi_{\{1\}}\, d\nu = \nu(\{1\}) = 1$. It follows that

$$\iint f(m,n)\, d\nu(n)\, d\mu(m) = \int \chi_{\{1\}}\, d\mu = \mu(\{1\}) = 1.$$

49. (a) Since μ and ν are σ-finite,

$$\int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y) = (\mu\times\nu)(E) = 0.$$

This implies that $\nu(E_x) = \mu(E^y) = 0$ for almost every $x \in X$ and $y \in Y$.

(b) Let $E \subseteq X \times Y$ be a null set such that $f(x,y) = 0$ for all $x \in X$ and $y \in Y$ such that $(x,y) \notin E$. If $x \in X$, then $f_x(y) = 0$ for all $y \in Y$ such that $y \notin E_x$. Hence $f_x = 0$ almost everywhere, so f_x is integrable with $\int f_x\, d\nu = 0$, for almost all $x \in X$ (by the previous exercise). Similarly f^y is integrable and $\int f^y\, d\mu = 0$ for almost every $y \in Y$.

Now let f be $\mathcal{L}$-measurable. There exists an $(\mathcal{M}\otimes\mathcal{N})$-measurable function g such that $f = g$ λ-almost everywhere. Moreover g_x is $\mathcal{N}$-measurable and g^y is $\mathcal{M}$-measurable for all $x \in X$ and $y \in Y$. If $f \geq 0$ then $g \geq 0$ without loss of generality, so by Tonelli's theorem $x \mapsto \int g_x\, d\nu$ and $y \mapsto \int g^y\, d\mu$ are non-negative and $(\mathcal{M}\otimes\mathcal{N})$-measurable, while

$$\int g\, d\lambda = \iint g(x,y)\, d\mu(x)\, d\nu(y) = \iint g(x,y)\, d\nu(y)\, d\mu(x). \tag{2}$$

Since $|g| = |f|$ λ-almost everywhere, $\int |g|\, d(\mu\times\nu) = \int |g|\, d\lambda = \int |f|\, d\lambda$ and hence $g \in L^1(\mu\times\nu)$ whenever $f \in L^1(\lambda)$. By Fubini's theorem, this implies that $g_x \in L^1(\nu)$ and $g^y \in L^1(\mu)$ for almost all $x \in X$ and $y \in Y$, while $x \mapsto \int g_x\, d\nu$ and $y \mapsto \int g^y\, d\mu$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Also (2) holds in this case. The corresponding statements about f follow by applying part (b) of this exercise to $f - g$. In particular, $f_x - g_x = 0$ almost everywhere for almost all $x \in X$, so f_x is $\mathcal{N}$-measurable for almost all $x \in X$. Similarly f^y is $\mathcal{M}$-measurable for almost all $y \in Y$. Since $f_x - g_x \in L^1(\nu)$ and $f^y - g^y \in L^1(\mu)$ for almost all $x \in X$ and $y \in Y$, it is clear that $f_x \in L^1(\nu)$ and $f^y \in L^1(\mu)$ for almost all $x \in X$

and $y \in Y$, provided that $f \in L^1(\lambda)$. In either of the two cases $\int g_x \, d\nu = \int (f_x - g_x) \, d\nu + \int g_x \, d\nu = \int f_x \, d\nu$ for almost all $x \in X$, so $x \mapsto \int f_x \, d\nu$ is measurable and in the second case, integrable (for the first case, assume without loss of generality that $g \leq f$). The same clearly holds for $y \mapsto \int f^y \, d\mu$, so (because $f = g$ almost everywhere)

$$\int f \, d\lambda = \iint g(x,y) \, d\mu(x) \, d\nu(y) = \iint f(x,y) \, d\mu(x) \, d\nu(y) = \iint g(x,y) \, d\nu(y) \, d\mu(x) = \iint f(x,y) \, d\nu(y) \, d\mu(x).$$

50. Subtraction is a continuous map from $[0,\infty] \times [0,\infty) \to (-\infty, \infty]$, since it is constant on the closed set $\{\infty\} \times [0,\infty)$. In particular, the preimage of $[0,\infty)$ (or $(0,\infty)$) under subtraction is an open subset of $[0,\infty] \times [0,\infty)$, so it is a countable union of rectangles $(A_n \times B_n)_{n=1}^\infty$. Hence, the preimage of $[0,\infty)$ (or $(0,\infty)$) under the map $(x,y) \mapsto f(x) - y$ is

$$\begin{aligned} E &:= \{(x,y) \in X \times [0,\infty) \mid (f(x),y) \in A_n \times B_n \text{ for some } n \in \mathbb{N}\} \\ &= \cup_{n=1}^\infty \{(x,y) \in X \times [0,\infty) \mid x \in f^{-1}(A_n) \text{ and } y \in B_n\} \\ &= \cup_{n=1}^\infty (f^{-1}(A_n) \times B_n). \end{aligned}$$

Clearly E is $(\mathcal{M} \otimes \mathcal{B}_{\overline{\mathbb{R}}})$-measurable, and hence $G_f = E \cup (f^{-1}(\{\infty\}) \times \{\infty\})$ is also $(\mathcal{M} \otimes \mathcal{B}_{\overline{\mathbb{R}}})$-measurable (the same holds for the redefinition of G_f, because in that case $G_f = E$ if we take $(0,\infty)$ instead of $[0,\infty)$ above). For the second part, assume that $m(\{\infty\}) = 0$ and $m|_{[0,\infty)}$ agrees with the Lebesgue measure. By Tonelli's theorem

$$(\mu \times m)(G_f) = (\mu \times m)(E) = \int \chi_E = \iint \chi_E(x,y) \, dm(y) \, d\mu(x).$$

If $x \in X$ then $(x,y) \in E$ iff $y \in [0,\infty)$ and $f(x) - y \geq 0$ (or > 0), so $\int \chi_E(x,y) \, dm(y) = \int_0^{f(x)} 1 \, dm(y) = f(x)$. Therefore $(\mu \times m)(G_f) = \int f \, d\mu$ as required.

51. (a) Define $F, G : X \times Y \to \mathbb{C}$ by $F(x,y) := f(x)$ and $G(x,y) := g(y)$. Then $F^{-1}(A) = f^{-1}(A) \times Y$ and $G^{-1}(A) = X \times g^{-1}(A)$ for all $A \subseteq \mathbb{C}$, so F and G are $(\mathcal{M} \otimes \mathcal{N})$-measurable. Therefore $h = FG$ is $(\mathcal{M} \otimes \mathcal{N})$-measurable.

(b) Suppose $f \geq 0$ and $g \geq 0$. There exist increasing sequences $(\phi_n)_{n=1}^\infty$ and $(\psi_n)_{n=1}^\infty$ of non-negative simple functions which converge pointwise to f and g respectively. For each $n \in \mathbb{N}$ define $\Phi_n, \Psi_n : X \times Y \to [0,\infty]$ as in part (a), so that $(\Phi_n \Psi_n)_{n=1}^\infty$ converges pointwise to h. Fix $n \in \mathbb{N}$, and write $\phi_n = \sum_{i=1}^k a_i \chi_{A_i}$ and $\phi_n = \sum_{j=1}^l b_j \chi_{B_j}$ for some $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l \in [0,\infty]$ and measurable sets $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l$. Clearly

$$\Phi_n \Psi_n = \left(\sum_{i=1}^k a_i \chi_{(A_i \times Y)} \right) \left(\sum_{j=1}^l b_j \chi_{(X \times B_j)} \right) = \sum_{i=1}^k \sum_{j=1}^l a_i \chi_{(A_i \times Y)} b_j \chi_{(X \times B_j)} = \sum_{i=1}^k \sum_{j=1}^l a_i b_j \chi_{(A_i \times B_j)},$$

and hence

$$\int \Phi_n \Psi_n = \sum_{i=1}^k \sum_{j=1}^l a_i b_j (\mu \times \nu)(A_i \times B_j) = \sum_{i=1}^k \sum_{j=1}^l a_i \mu(A_i) b_j \nu(B_j) = \left(\sum_{i=1}^k a_i \mu(A_i) \right) \left(\sum_{j=1}^l b_j \nu(B_j) \right) = \int \phi_n \cdot \int \psi_n.$$

By the monotone convergence theorem, it follows that

$$\int h = \lim_{n \to \infty} \int \Phi_n \Psi_n = \lim_{n \to \infty} \int \phi_n \cdot \int \psi_n = \lim_{n \to \infty} \int \phi_n \cdot \lim_{n \to \infty} \int \psi_n = \int f \cdot \int g.$$

Hence, in general $\int |h| = \int |f| \cdot \int |g| < \infty$, so $h \in L^1(\mu \times \nu)$. If $f(X) \subseteq \mathbb{R}$ and $g(Y) \subseteq \mathbb{R}$, then

$$\int h = \int h^+ - \int h^-$$

$$\begin{aligned}
&= \int F^+G^+ + \int F^-G^- - \int F^+G^- - \int F^-G^+ \\
&= \int f^+ \cdot \int g^+ + \int f^- \cdot \int g^- - \int f^+ \cdot \int g^- - \int f^- \cdot \int g^+ \\
&= \int f^+ \left(\int g^+ - \int g^-\right) + \int f^- \left(\int g^- - \int g^+\right) \\
&= \left(\int f^+ - \int f^-\right)\left(\int g^+ - \int g^-\right) \\
&= \int f \cdot \int g.
\end{aligned}$$

Since $FG = (\operatorname{Re}(F)+i\operatorname{Im}(F))(\operatorname{Re}(G)+i\operatorname{Im}(G)) = \operatorname{Re}(F)\operatorname{Re}(G)-\operatorname{Im}(F)\operatorname{Im}(G)+i(\operatorname{Re}(F)\operatorname{Im}(G)+\operatorname{Im}(F)\operatorname{Re}(G))$,

$$\begin{aligned}
\int h &= \int \operatorname{Re}(h) + i\int \operatorname{Im}(h) \\
&= \int \operatorname{Re}(F)\operatorname{Re}(G) - \int \operatorname{Im}(F)\operatorname{Im}(G) + i\int \operatorname{Re}(F)\operatorname{Im}(G) + i\int \operatorname{Im}(F)\operatorname{Re}(G) \\
&= \int \operatorname{Re}(f)\cdot\int \operatorname{Re}(g) - \int \operatorname{Im}(f)\cdot\int \operatorname{Im}(g) + i\int \operatorname{Re}(f)\cdot\int \operatorname{Im}(g) + i\int \operatorname{Im}(f)\cdot\int \operatorname{Re}(g) \\
&= \int \operatorname{Re}(f)\left(\int \operatorname{Re}(g) + i\int \operatorname{Im}(g)\right) - \int \operatorname{Im}(f)\left(\int \operatorname{Im}(g) - i\int \operatorname{Re}(g)\right) \\
&= \int \operatorname{Re}(f)\left(\int \operatorname{Re}(g) + i\int \operatorname{Im}(g)\right) + i\int \operatorname{Im}(f)\left(\int \operatorname{Re}(g) + i\int \operatorname{Im}(g)\right) \\
&= \left(\int \operatorname{Re}(f) + i\int \operatorname{Im}(f)\right)\left(\int \operatorname{Re}(g) + i\int \operatorname{Im}(g)\right) \\
&= \int f \cdot \int g.
\end{aligned}$$

55. (a) Fix $y \in (0,1]$, and define $F : [0,1] \to \mathbb{R}$ by $F(x) \coloneqq x(x^2+y^2)^{-1}$. By the quotient rule

$$F'(x) = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = -f^y(x)$$

for all $x \in [0,1]$. This implies that

$$\int_0^1 (f^y)^- = \int_0^y -f^y = F(y) - F(0) = F(y) = \frac{y}{2y^2} = \frac{1}{2y}$$

and similarly $\int_0^1 (f^y)^+ = \int_y^1 f^y = -F(1)+F(y) = \frac{1}{2y} - \frac{1}{1+y^2}$. By the Tonelli and monotone convergence theorems

$$\int_E f^- = \int_0^1\int_0^1 f^-(x,y)\,dx\,dy = \int_0^1 \frac{1}{2y}\,dy = \lim_{n\to\infty}\int_{\frac{1}{n}}^1 \frac{1}{2y}\,dy = \lim_{n\to\infty}\frac{-\log(\frac{1}{n})}{2} = \infty,$$

which implies that

$$\int_E f^+ = \int_0^1\int_0^1 f^+(x,y)\,dx\,dy = \int_0^1 \left(\frac{1}{2y} - \frac{1}{1+y^2}\right)dy = \infty$$

because $\int_0^1 \frac{1}{1+y^2}\,dy \le \int_0^1 1\,dy < \infty$ and

$$\int_0^1 \left(\frac{1}{2y} - \frac{1}{1+y^2}\right)dy + \int_0^1 \frac{1}{1+y^2}\,dy = \int_0^1 \frac{1}{2y}\,dy = \infty$$

This shows that $\int_E f$ is not defined. However,

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 (-F(1)+F(0))\,dy = -\int_0^1 \frac{1}{1+y^2}\,dy = -(\tan^{-1}(1)-\tan^{-1}(0)) = -\frac{\pi}{4}.$$

Since $f(x,y) = -f(y,x)$ for all $x, y \in (0,1]$, it follows that

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{\pi}{4}.$$

(b) Since $f(x,y) \geq 0$ for all $(x,y) \in E \setminus \{(1,1)\}$, all three integrals exist and Tonelli's theorem implies that they are equal.

(c) By Tonelli's theorem, the fundamental theorem of calculus and the monotone convergence theorem,

$$\begin{aligned}
\int_E f^+ &= \int_0^1 \int_0^1 f^+(x,y)\,dx\,dy \\
&= \int_0^{\frac{1}{2}} \int_{\frac{1}{2}+y}^1 \left(x - \frac{1}{2}\right)^{-3} dx\,dy \\
&= \int_0^{\frac{1}{2}} (-2)^{-1} \left(\left(\frac{1}{2}\right)^{-2} - y^{-2}\right) dy \\
&= \frac{1}{2} \int_0^{\frac{1}{2}} (y^{-2} - 4)\,dy \\
&= \lim_{n\to\infty} \frac{1}{2} \int_{\frac{1}{n+4}}^{\frac{1}{2}} (y^{-2} - 4)\,dy \\
&= \lim_{n\to\infty} \frac{1}{2} \left(-\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{n+4}\right)^{-1} - \frac{4}{2} + \frac{4}{n+4}\right) \\
&= \lim_{n\to\infty} \frac{n}{2} \\
&= \infty.
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_E f^- &= \int_0^1 \int_0^1 f^-(x,y)\,dx\,dy \\
&= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-y} \left(\frac{1}{2} - x\right)^{-3} dx\,dy \\
&= \int_0^{\frac{1}{2}} 2^{-1} \left(y^{-2} - \left(\frac{1}{2}\right)^{-2}\right) dy \\
&= \frac{1}{2} \int_0^{\frac{1}{2}} (y^{-2} - 4)\,dy \\
&= \infty.
\end{aligned}$$

Therefore $\int_E f$ does not exist. However, the above working implies that

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^{\frac{1}{2}} \int_0^1 f(x,y)\,dx\,dy$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2}} \left(\int_0^1 f^+(x,y)\,dx - \int_0^1 f^-(x,y)\,dx \right) dy \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{2}(y^{-2}-4) - \frac{1}{2}(y^{-2}-4) \right) dy \\
&= 0.
\end{aligned}
$$

If $x \in [0, \frac{1}{2}]$, then $f_x \leq 0$ and hence

$$\int_0^1 f(x,y)\,dy = -\int_0^1 f^-(x,y)\,dy = -\int_0^{\frac{1}{2}-x} \left(\frac{1}{2}-x\right)^{-3} dy = -\left(\frac{1}{2}-x\right)^{-2}.$$

Similarly, if $x \in [\frac{1}{2}, 1]$ then

$$\int_0^1 f(x,y)\,dy = \int_0^1 f^+(x,y)\,dy = \int_0^{x-\frac{1}{2}} \left(x-\frac{1}{2}\right)^{-3} dy = \left(x-\frac{1}{2}\right)^{-2}.$$

By the monotone convergence theorem and the fundamental theorem of calculus, this implies that

$$
\begin{aligned}
\int_0^1 \left(\int_0^1 f(x,y)\,dy \right)^+ dx &= \int_{\frac{1}{2}}^1 \left(x-\frac{1}{2}\right)^{-2} dx \\
&= \lim_{n\to\infty} \int_{\frac{1}{2}+\frac{1}{n+2}}^1 \left(x-\frac{1}{2}\right)^{-2} dx \\
&= \lim_{n\to\infty} \left(-\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{n+2}\right)^{-1} \right) \\
&= \lim_{n\to\infty} n \\
&= \infty.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\int_0^1 \left(\int_0^1 f(x,y)\,dy \right)^- dx &= \int_0^{\frac{1}{2}} \left(\frac{1}{2}-x\right)^{-2} dx \\
&= \lim_{n\to\infty} \int_0^{\frac{1}{2}-\frac{1}{n+2}} \left(\frac{1}{2}-x\right)^{-2} dx \\
&= \lim_{n\to\infty} \left(\left(\frac{1}{n+2}\right)^{-1} - \left(\frac{1}{2}\right)^{-1} \right) \\
&= \lim_{n\to\infty} n \\
&= \infty.
\end{aligned}
$$

This shows that $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ does not exist.

56. Define a measurable function $h : (0,a)^2 \to \mathbb{C}$ by $h(t,x) := t^{-1} f(t) \chi_E(t,x)$, where $E := \{(t,x) \in (0,a)^2 \mid x < t\}$ is open, hence measurable. Then $g(x) = \int_0^a t^{-1} f(t) \chi_{(x,a)}(t)\,dt = \int h_x$ for all $x \in (0,a)$. By Tonelli's theorem

$$\int |h| = \int_0^a \int_0^a t^{-1} |f(t)| \chi_E(t,x)\,dx\,dt = \int_0^a \int_0^t t^{-1} |f(t)|\,dx\,dt = \int_0^a |f(t)|\,dt < \infty.$$

Hence h is integrable, so g is integrable by Fubini's theorem, and

$$\int_0^a g = \int_0^a \int_0^a h_x(t)\,dt\,dx = \int h = \int_0^a \int_0^a t^{-1} f(t)\chi_E(t,x)\,dx\,dt = \int_0^a \int_0^t t^{-1} f(t)\,dx\,dt = \int_0^a f(t)\,dt.$$

58. Let $s \in (0,\infty)$ and define $f : [0,\infty) \times [0,1] \to [0,1]$ by $f(x,y) := e^{-sx}\sin(2xy)$. Clearly $|f(x,y)| \leq e^{-sx}$ for all $x, y \in [0,\infty) \times [0,1]$, so $f \in L^1$. Since

$$\int_0^\infty \int_0^1 e^{-sx}\sin(2xy)\,dy\,dx = \int_0^\infty \left(-e^{-sx}\frac{\cos(2xy)}{2x}\right)\Big|_0^1 dx = \int_0^\infty e^{-sx}\frac{1-\cos(2x)}{2x}\,dx = \int_0^\infty e^{-sx}x^{-1}\sin^2(x)\,dx,$$

Fubini's theorem implies that

$$\int_0^\infty e^{-sx}x^{-1}\sin^2(x)\,dx = \int_0^1 \int_0^\infty e^{-sx}\sin(2xy)\,dx\,dy = \int_0^1 \frac{2y}{s^2+4y^2}\,dy = \frac{1}{4}\log(4^{-1}s^2+y^2)\Big|_0^1 = \frac{1}{4}\log(1+4s^{-2})$$

(the middle step can be done using integration by parts or by expressing sin as a difference of complex exponentials).

59. (a) Let $n \in \mathbb{N}$. If $x \in [(n+\frac{1}{6})\pi, (n+\frac{5}{6})\pi]$ then $|\sin(x)| \geq \frac{1}{2}$, and $x^{-1} \geq (n+1)^{-1}\pi^{-1}$. Therefore

$$\int_0^\infty |f| \geq \int \sum_{n=1}^\infty \frac{1}{2(n+1)\pi}\chi_{[(n+\frac{1}{6})\pi,(n+\frac{5}{6})\pi]} = \sum_{n=1}^\infty \int \frac{1}{2(n+1)\pi}\chi_{[(n+\frac{1}{6})\pi,(n+\frac{5}{6})\pi]} = \sum_{n=1}^\infty \frac{\frac{4}{6}\pi}{2(n+1)\pi} = \infty,$$

by the monotone convergence theorem.

(b) Fix $b \in (0,\infty)$, and define $f : (0,b)^2 \to \mathbb{R}$ by $f(x,y) := e^{-xy}\sin(x)$. Clearly $|f| \leq 1$, so $\int |f| \leq b^2 < \infty$. Hence, by Fubini's theorem and the fundamental theorem of calculus

$$\begin{aligned}\int_0^b \int_0^b f(x,y)\,dx\,dy &= \int_0^b \int_0^b e^{-xy}\sin(x)\,dy\,dx \\ &= \int_0^b \left(\frac{e^{-xb}\sin(x)}{-x} - \frac{e^0\sin(x)}{-x}\right)dx \\ &= \int_0^b \left(\frac{\sin(x)}{x} - \frac{e^{-xb}\sin(x)}{x}\right)dx \qquad (3)\end{aligned}$$

Since $|\sin(x)| \leq x$ for all $x \in (0,\infty)$ it is clear that

$$\left|\int_0^b \frac{e^{-xb}\sin(x)}{x}\,dx\right| \leq \int_0^b \left|\frac{e^{-xb}\sin(x)}{x}\right| dx \leq \int_0^b e^{-xb}\,dx = \frac{e^{-b^2}}{-b} - \frac{e^0}{-b} = \frac{1-e^{-b^2}}{b}. \qquad (4)$$

Integrating f by parts twice with respect to x suggests we define a function $F : (0,b)^2 \to \mathbb{R}$ by

$$F(x,y) := -e^{-xy}\frac{y\sin(x)+\cos(x)}{y^2+1},$$

so that

$$\begin{aligned}\frac{\partial}{\partial x}F(x,y) &= ye^{-xy}\frac{y\sin(x)+\cos(x)}{y^2+1} - e^{-xy}\frac{y\cos(x)-\sin(x)}{y^2+1} \\ &= e^{-xy}\frac{y^2\sin(x)+y\cos(x)-y\cos(x)+\sin(x)}{y^2+1} \\ &= f(x,y)\end{aligned}$$

and hence

$$\begin{aligned}
\int_0^b \int_0^b f(x,y)\,dx\,dy &= \int_0^b (F(b,y) - F(0,y))\,dy \\
&= \int_0^b \left(e^0 \frac{y\sin(0)+\cos(0)}{y^2+1} - e^{-by}\frac{y\sin(b)+\cos(b)}{y^2+1} \right) dy \\
&= \int_0^b \left(\frac{1}{y^2+1} - e^{-by}\frac{y\sin(b)+\cos(b)}{y^2+1} \right) dy. \qquad (5)
\end{aligned}$$

Either $y \le 1$ or $y \le y^2$ for all $y \in (0,\infty)$, so

$$\left| \int_0^b e^{-by}\frac{y\sin(b)+\cos(b)}{y^2+1}\,dy \right| \le \int_0^b e^{-by}\left(\frac{y|\sin(b)|}{y^2+1} + \frac{|\cos(b)|}{y^2+1} \right) dy \le \int_0^b 2e^{-by}\,dy = 2\frac{1-e^{-b^2}}{b}. \qquad (6)$$

Together (3), (4), (5) and (6) imply that

$$\lim_{b\to\infty} \int_0^b \frac{\sin(x)}{x}\,dx + 0 = \lim_{b\to\infty} \int_0^b \int_0^b f(x,y)\,dx\,dy = \lim_{b\to\infty} \int_0^b \frac{1}{y^2+1}\,dy + 0 = \lim_{b\to\infty} (\tan^{-1}(b) - \tan^{-1}(0)) = \frac{\pi}{2}.$$

60. If $x, y \in (0,\infty)$ then, by Exercise 51

$$\Gamma(x)\Gamma(y) = \int_0^\infty s^{x-1}e^{-s}\,ds \int_0^\infty t^{y-1}e^{-t}\,dt = \int_0^\infty \int_0^\infty s^{x-1}t^{y-1}e^{-(s+t)}\,ds\,dt.$$

Define $G : (0,\infty)\times(0,1) \to (0,\infty)^2$ by $G(u,v) := (uv, u(1-v))$, and check that G is a C^1-diffeomorphism with Jacobian determinant $-u$ at the point (u,v). By Theorem 2.47, Tonelli's theorem and Exercise 51, $\Gamma(x)\Gamma(y)$ is

$$\int_0^1 \int_0^\infty (uv)^{x-1}(u(1-v))^{y-1}e^{-u}u\,du\,dv = \int_0^1 v^{x-1}(1-v)^{y-1}\,dv \int_0^\infty u^{x+y-1}e^{-u}\,du = \Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

61. (a) Let $\alpha, \beta \in (0,\infty)$ and $x \in [0,\infty)$. Note that

$$\begin{aligned}
I_\alpha(I_\beta f)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} I_\beta f(t)\,dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s)\,ds\,dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x \int_0^t (x-t)^{\alpha-1}(t-s)^{\beta-1}f(s)\,ds\,dt.
\end{aligned}$$

Since f is bounded on $[0,x]$ and $\int_0^1 t^\gamma\,dt < \infty$ for all $\gamma \in (-1,\infty)$, the Fubini-Tonelli theorem implies that

$$I_\alpha(I_\beta f)(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x \int_s^x (x-t)^{\alpha-1}(t-s)^{\beta-1}f(s)\,dt\,ds.$$

For each s we apply the substitution $u := (t-s)/(x-s)$ to the inner integral, and obtain

$$\begin{aligned}
I_\alpha(I_\beta f)(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x \int_0^1 (x-s-u(x-s))^{\alpha-1}(u(x-s))^{\beta-1}f(s)(x-s)\,du\,ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x \int_0^1 (x-s)^{\alpha-1}(1-u)^{\alpha-1}u^{\beta-1}(x-s)^{\beta-1}f(s)(x-s)\,du\,ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x (x-s)^{\alpha+\beta-1}f(s)\,ds \int_0^1 (1-u)^{\alpha-1}u^{\beta-1}\,du \\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_0^x (x-s)^{\alpha+\beta-1}f(s)\,ds \\
&= I_{\alpha+\beta}f(x).
\end{aligned}$$

(b) Clearly $I_1 f$ is an antiderivative of f. Given $n \in \mathbb{N}$, we aim to show that $I_n f$ is an nth-order antiderivative of f. By induction, we may assume that $n > 1$ and that $I_{n-1}(I_1 f)$ is an $(n-1)$th-order antiderivative of $I_1 f$. Hence

$$(I_n f)^{(n)} = ((I_{n-1}(I_1 f))^{(n-1)})' = (I_1 f)' = f.$$

62. Let $E \subseteq S^{n-1}$ be measurable and $T \in \mathrm{SO}(n)$ a rotation. We want to show that $\sigma(T(E)) = \sigma(E)$. Note that

$$\Phi^{-1}((0,1) \times E) = \left\{ x \in \mathbb{R}^n \setminus \{0\} \;\middle|\; |x| \in (0,1) \text{ and } \frac{x}{|x|} \in E \right\},$$

while

$$\begin{aligned}\Phi^{-1}((0,1) \times T(E)) &= \left\{ x \in \mathbb{R}^n \setminus \{0\} \;\middle|\; |x| \in (0,1) \text{ and } \frac{x}{|x|} \in T(E) \right\} \\ &= \left\{ x \in \mathbb{R}^n \setminus \{0\} \;\middle|\; |T^{-1}x| \in (0,1) \text{ and } \frac{T^{-1}x}{|T^{-1}x|} \in E \right\} \\ &= \{ x \in \mathbb{R}^n \setminus \{0\} \mid T^{-1}x \in \Phi^{-1}((0,1) \times E)\} \\ &= T(\Phi^{-1}((0,1) \times E)).\end{aligned}$$

Therefore

$$\rho((0,1))\sigma(E) = m_*((0,1)\times E) = m(\Phi^{-1}((0,1)\times E)) = m(T(\Phi^{-1}((0,1)\times E))) = m_*((0,1)\times T(E)) = \rho((0,1))\sigma(T(E)).$$

Since $\rho((0,1)) = \int_0^1 r^{n-1}\,dr = n^{-1} > 0$, it follows that $\sigma(T(E)) = \sigma(E)$.

64. Let $a, b \in \mathbb{R}$ and set $\alpha \coloneqq a + n - 1$. By Corollary 2.51 it suffices to determine when

$$\int_0^{1/2} r^\alpha |\log(r)|^b\,dr \quad \text{and} \quad \int_2^\infty r^\alpha |\log(r)|^b\,dr$$

are finite. We claim that, given $\varepsilon \in (0,\infty)$, there exists $\delta \in (0, \frac{1}{2})$ such that $|\log(r)| \le 2r^{-\varepsilon}$ for all $r \in (0,\delta)$. To prove this, choose $n \in \mathbb{N}$ such that $n^{-1} < \varepsilon$ and define $\delta \coloneqq \log(n!)^{-n}$. If $r \in (0,\delta)$ then

$$|\log(r)| = \log(r^{-1}) = \log((r^{-1/n})^n) \le \log(n!\,e^{r^{-1/n}}) = \log(n!) + r^{-1/n} = \delta^{-1/n} + r^{-1/n} \le 2r^{-1/n} \le 2r^{-\varepsilon},$$

as claimed. If $\alpha > -1$ and $b > 0$, set $\varepsilon \coloneqq \frac{1}{2b}(\alpha+1)$, take the corresponding δ and note that

$$\int_0^\delta r^\alpha |\log(r)|^b\,dr \le \int_0^\delta r^\alpha 2^b r^{-(\alpha+1)/2}\,dr = 2^b \int_0^\delta r^{(\alpha-1)/2}\,dr < \infty,$$

as $\frac{1}{2}(\alpha - 1) > -1$. Since $r^\alpha|\log(r)|^b$ is bounded for $r \in (\delta, \frac{1}{2})$, it follows that the first integral is finite. If $\alpha > -1$ and $b \le 0$ then $|\log(r)|^b = \log(r^{-1})^b \le \log(2)^b$ for all $r \in (0, \frac{1}{2})$, in which case the first integral is finite. Similarly, it is infinite if $\alpha < -1$ and $b > 0$. If $\alpha < -1$ and $b < 0$, set $\varepsilon \coloneqq -\frac{1}{2b}(\alpha+1)$, take the corresponding δ and note that

$$\int_0^\delta r^\alpha |\log(r)|^b\,dr \ge \int_0^\delta r^\alpha 2^b r^{(\alpha+1)/2}\,dr = 2^b \int_0^\delta r^{(3\alpha+1)/2}\,dr = \infty,$$

as $\frac{1}{2}(3\alpha+1) < -1$. Thus, the first integral is finite. Finally, set $\alpha \coloneqq -1$. By the monotone convergence theorem

$$\int_0^{1/2} r^\alpha |\log(r)|^b\,dr = -\lim_{t\to 0} \int_t^{1/2} (-\log(r))^b\,d(-\log(r)) = -\lim_{t\to 0} \left.\frac{(-\log(r))^{b+1}}{b+1}\right|_t^{1/2},$$

which is finite iff $b < -1$ (the case $b = -1$ should really be handled separately). In summary, $\int_0^{1/2} r^\alpha |\log(r)|^b \, dr$ is finite iff $a > -n$ or ($a = -n$ and $b < -1$). The second integral can be treated similarly; alternatively we can substitute $s := r^{-1}$ and note that

$$\int_2^\infty r^\alpha |\log(r)|^b \, dr = \int_0^{1/2} s^{-\alpha-2} |\log(s)|^b \, ds,$$

which is finite iff $-\alpha - 2 > -1$ or ($-\alpha - 2 = -1$ and $b < -1$); in other words $a < -n$ or ($a = -n$ and $b < -1$).

65. (a) This is clear if $n = 2$. If $n \geq 3$, let $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the map corresponding to G. Also let $\pi : \mathbb{R}^{n-1} \to \mathbb{R}^{n-2}$ and $\rho : \mathbb{R}^{n-1} \to \mathbb{R}$ be projections on to the first $n - 2$ and last coordinate(s), respectively. Clearly

$$G(r, \phi_1, \ldots, \phi_{n-2}, \theta) = (\pi(F(r, \phi_1, \ldots, \phi_{n-2})), \rho(F(r, \phi_1, \ldots, \phi_{n-2})) \cos(\theta), \rho(F(r, \phi_1, \ldots, \phi_{n-2})) \sin(\theta)).$$

If $x \in \mathbb{R}^n$ then (by induction) we may assume that $(x_1, \ldots, x_{n-2}, |(x_{n-1}, x_n)|) = F(r, \phi_1, \ldots, \phi_{n-2})$ for some $r, \phi_1, \ldots, \phi_{n-2} \in \mathbb{R}$, in which case it is clear that $x = G(r, \phi_1, \ldots, \phi_{n-2}, \theta)$ for some $\theta \in \mathbb{R}$. Moreover, if $r, \phi_1, \ldots, \phi_{n-2}, \theta \in \mathbb{R}$ then (assuming, by induction, that $|F(r, \phi_1, \ldots, \phi_{n-2})| = |r|$)

$$\begin{aligned}
|G(r, \phi_1, \ldots, \phi_{n-2}, \theta)| &= \sqrt{|\pi(F(r, \phi_1, \ldots, \phi_{n-2}))|^2 + \rho(F(r, \phi_1, \ldots, \phi_{n-2}))^2 \cos^2(\theta) + \rho(F(r, \phi_1, \ldots, \phi_{n-2}))^2 \sin^2(\theta)} \\
&= \sqrt{|r|^2 - \rho(F(r, \phi_1, \ldots, \phi_{n-2}))^2 + \rho(F(r, \phi_1, \ldots, \phi_{n-2}))^2 (\cos^2(\theta) + \sin^2(\theta))} \\
&= \sqrt{r^2} \\
&= |r|.
\end{aligned}$$

(b) Denote the component functions of F by $F^1, \ldots, F^{n-1}$. The Jacobian of G at a point $(r, \phi_1, \ldots, \phi_{n-2}, \theta) \in \mathbb{R}^n$ is

$$\begin{pmatrix}
F_r^1 & F_{\phi_1}^1 & \cdots & F_{\phi_{n-2}}^1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_r^{n-2} & F_{\phi_1}^{n-2} & \cdots & F_{\phi_{n-2}}^{n-2} & 0 \\
F_r^{n-1} \cos(\theta) & F_{\phi_1}^{n-1} \cos(\theta) & \cdots & F_{\phi_{n-2}}^{n-1} \cos(\theta) & -F^{n-1} \sin(\theta) \\
F_r^{n-1} \sin(\theta) & F_{\phi_1}^{n-1} \sin(\theta) & \cdots & F_{\phi_{n-2}}^{n-1} \sin(\theta) & F^{n-1} \cos(\theta)
\end{pmatrix},$$

so its determinant is $F^{n-1} \sin^2(\theta) \det(DF) + F^{n-1} \cos^2(\theta) \det(DF) = r \sin(\phi_1) \ldots \sin(\phi_{n-2}) \det(DF)$. It easily follows by induction that $\det(DG)$ has the required form (the case $n = 2$ is trivial).

(c) This is well-known for $n = 2$, so we may assume that $n \geq 3$ and $F|_{(0,\infty)\times(0,\pi)^{n-3}\times(0,2\pi)}$ is injective. We may refine our argument from part (a) to show that $G(\Omega)$ contains the points of $\mathbb{R}^n$ whose coordinates are all nonzero, in which case $\mathbb{R}^n \setminus G(\Omega)$ is clearly a null set. If $(r, \phi_1, \ldots, \phi_{n-2}, \theta) \in \Omega$ and $(R, \Phi_1, \ldots, \Phi_{n-2}, \Theta) \in \Omega$ map to the same point under G, then

$$\pi(F(r, \phi_1, \ldots, \phi_{n-2})) = \pi(F(R, \Phi_1, \ldots, \Phi_{n-2}))$$

and $\rho(F(r, \phi_1, \ldots, \phi_{n-2}))^2 = \rho(F(R, \Phi_1, \ldots, \Phi_{n-2}))^2$ (because $\cos^2(\theta) + \sin^2(\theta) = \cos^2(\Theta) + \sin^2(\Theta)$). By definition $\rho(F(r, \phi_1, \ldots, \phi_{n-2})) = r \sin(\phi_1) \ldots \sin(\phi_{n-2})$, which is positive by the definition of Ω. Therefore

$$\rho(F(r, \phi_1, \ldots, \phi_{n-2})) = \rho(F(R, \Phi_1, \ldots, \Phi_{n-2})),$$

and hence $(r, \phi_1, \ldots, \phi_{n-2}) = (R, \Phi_1, \ldots, \Phi_{n-2})$. It clearly follows that $\theta = \Theta$. This shows that $G|_\Omega$ is injective, so it has an inverse defined on $G(\Omega)$; the inverse function theorem (cf. part (b)) implies that the inverse is smooth and $G(\Omega)$ is open, in which case $G|_\Omega$ is a diffeomorphism.

(d) If we view S^{n-1} as a smooth manifold it is straightforward to show that $(F|_{\Omega'})^{-1}$ is a diffeomorphism, but that's outside the scope of this course. Given an integrable function $f : S^1 \to \mathbb{C}$, define $g : \mathbb{R}^n \to \mathbb{C}$ by $g(x) := f(\frac{x}{|x|})\chi_{B_1(0)}(x)$. By Theorem 2.49 and Exercise 51

$$\int_{\mathbb{R}^n} g = \int_0^\infty \int_{S^{n-1}} g(rx)r^{n-1}\,d\sigma(x)\,dr = \int_0^1 \int_{S^{n-1}} f(x)r^{n-1}\,d\sigma(x)\,dr = \frac{1}{n}\int_{S^{n-1}} f.$$

On the other hand,

$$\begin{aligned}
&\int_{\Omega'} f(F(\phi_1,\ldots,\phi_{n-2},\theta))\sin^{n-2}(\phi_1)\ldots\sin(\phi_{n-2})\,d\phi_1\cdots d\phi_{n-2}\,d\theta \\
&= n\int_0^1 r^{n-1}\,dr \int_{\Omega'} f(F(\phi_1,\ldots,\phi_{n-2},\theta))|\sin^{n-2}(\phi_1)\ldots\sin(\phi_{n-2})|\,d\phi_1\cdots d\phi_{n-2}\,d\theta \\
&= n\int_0^\infty \int_{\Omega'} g(rF(\phi_1,\ldots,\phi_{n-2},\theta))r^{n-1}|\sin^{n-2}(\phi_1)\ldots\sin(\phi_{n-2})|\,d\phi_1\cdots d\phi_{n-2}\,d\theta\,dr \\
&= n\int_{\Omega} g(G(r,\phi_1,\ldots,\phi_{n-2},\theta))|\det(D_{(r,\phi_1,\ldots,\phi_{n-2})}G)|\,d\phi_1\cdots d\phi_{n-2}\,d\theta\,dr \\
&= n\int_{G(\Omega)} g \\
&= \int_{S^{n-1}} f.
\end{aligned}$$