1. Suppose $(E_n)_{n=1}^\infty$ is an increasing sequence in $\mathcal{M}$. For each $n \in \mathbb{N}$ define $F_n := E_n \setminus E_{n-1}$ (with $E_0 := \varnothing$). Clearly

$$\nu(\cup_{n=1}^\infty E_n) = \nu(\cup_{n=1}^\infty F_n) = \sum_{n=1}^\infty \nu(F_n) = \lim_{N\to\infty} \sum_{n=1}^N \nu(F_n) = \lim_{N\to\infty} \nu(E_N).$$

If $(E_n)_{n=1}^\infty$ is a decreasing sequence in $\mathcal{M}$ and $\nu(E_1) < \infty$, then

$$\nu(\cap_{n=1}^\infty E_n) = \nu(E_1 \setminus (E_1 \setminus \cap_{n=1}^\infty E_n)) = \nu(E_1) - \nu(\cup_{n=1}^\infty (E_1 \setminus E_n)) = \nu(E_1) - \lim_{n\to\infty} \nu(E_1 \setminus E_n) = \lim_{n\to\infty} \nu(E_n).$$

2. Let E be measurable, and suppose that E is ν-null but $|\nu|(E) \neq 0$. Then $\nu^+(E) - \nu^-(E) = \nu(E) = 0$ and $\nu^+(E) + \nu^-(E) = |\nu|(E) > 0$, so $\nu^+(E) = \nu^-(E) > 0$. Since $\nu^+ \perp \nu^-$, there exist disjoint measurable sets A and B covering X such that A is ν^+-null and B is ν^--null. In particular $\nu^+(E \cap B) = \nu^+(E \cap B) + \nu^+(E \cap A) = \nu^+(E) > 0$ but $\nu^-(E \cap B) \leq \nu^-(B) = 0$. This implies that $\nu(E \cap B) > 0$, which is a contradiction because $E \cap B \subseteq E$ and E is ν-null.

Conversely, suppose that $|\nu|(E) = 0$. If $F \subseteq E$ is measurable, then $|\nu|(F) \leq |\nu|(E) = 0$ so $\nu^+(F) + \nu^-(F) = 0$. In particular $\nu^+(F) = 0 = \nu^-(F)$, so $\nu(F) = 0$ and hence E is ν-null.

Suppose that $\nu \perp \mu$, so that $X = A \cup B$ for some disjoint measurable sets A and B such that A is ν-null and B is μ-null. From above, $|\nu|(A) = 0$, so A is $|\nu|$-null (because $|\nu|$ is positive) and hence $|\nu| \perp \mu$.

Now suppose that $|\nu| \perp \mu$, so that $X = A \cup B$ for some disjoint measurable sets A and B such that A is $|\nu|$-null and B is μ-null. Since $\nu^+ \leq |\nu|$ and $\nu^- \leq |\nu|$ pointwise, A is ν^+-null and ν^--null. Therefore $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Finally, suppose that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Then $X = A \cup B = E \cup F$ for two pairs A, B and E, F of disjoint measurable sets such that A is ν^+-null, E is ν^--null and B and F are μ-null. Note that $B \cup F$ is μ-null, because every subset of $B \cup F = B \cup (F \setminus B)$ is the disjoint union of two μ-null sets. Moreover $X \setminus (B \cup F) = A \cap E$, which is both ν^+-null and ν^--null, and hence ν-null. This shows that $\nu \perp \mu$.

3. (a) Let $\phi \in L^+$ be a simple function, and write $\phi = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\int \phi \, d|\nu| = \sum_{i=1}^n a_i |\nu|(E_i) = \sum_{i=1}^n a_i(\nu^+(E_i) + \nu^-(E_i)) = \sum_{i=1}^n a_i \nu^+(E_i) + \sum_{i=1}^n a_i \nu^-(E_i) = \int \phi \, d\nu^+ + \int \phi \, d\nu^-. \quad (1)$$

Hence, if $f \in L^1(\nu)$ then

$$\int |f| \, d|\nu| = \sup\left\{\int \phi \, d\nu^+ + \int \phi \, d\nu^- \;\middle|\; \phi \in L^+ \text{ simple with } \phi \leq |f|\right\} \leq \int |f| \, d\nu^+ + \int |f| \, d\nu^- < \infty$$

because $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$. Therefore $L^1(\nu) \subseteq L^1(|\nu|)$. Conversely, if $f \in L^1(|\nu|)$ it is clear from (1) that $\int \phi \, d\nu^+ \leq \int \phi \, d|\nu| \leq \int |f| \, d|\nu|$ for all simple $\phi \in L^+$ with $\phi \leq |f|$, so $\int |f| \, d\nu^+ \leq \int |f| \, d|\nu| < \infty$. This shows that $L^1(|\nu|) \subseteq L^1(\nu^+)$, and a similar argument shows that $L^1(|\nu|) \subseteq L^1(\nu^-)$. Therefore $L^1(|\nu|) \subseteq L^1(\nu)$.

(b) Let A and B be disjoint measurable sets covering X such that A is ν^+-null and B is ν^--null. Then $g\chi_A = 0$ ν^+-a.e. and $g\chi_B = 0$ ν^--a.e. for every measurable function g. In particular, from (1) $\int \phi \, d|\nu| = \int \phi \, d\nu^+$ for all simple functions $\phi \in L^+$ with $\phi \leq f^+\chi_B$ or $\phi \leq f^-\chi_B$. This implies that

$$\int f\chi_B \, d|\nu| = \int f^+\chi_B \, d|\nu| - \int f^-\chi_B \, d|\nu| = \int f^+\chi_B \, d\nu^+ - \int f^-\chi_B \, d\nu^+ = \int f\chi_B \, d\nu^+,$$

and similarly $\int f\chi_A \, d|\nu| = \int f\chi_A \, d\nu^-$. Moreover $|\chi_A - \chi_B| = \chi_X$ because $A \cup B = X$ and $A \cap B = \varnothing$. Therefore

$$\left|\int f \, d\nu\right| = \left|\int f(\chi_A + \chi_B) \, d\nu^+ - \int f(\chi_A + \chi_B) \, d\nu^-\right|$$

$$
\begin{aligned}
&= \left| \int f\chi_B \, d\nu^+ - \int f\chi_A \, d\nu^- \right| \\
&= \left| \int f\chi_B \, d|\nu| - \int f\chi_A \, d|\nu| \right| \\
&= \left| \int f(\chi_B - \chi_A) \, d|\nu| \right| \\
&\le \int |f(\chi_B - \chi_A)| \, d|\nu| \\
&= \int |f| \, d|\nu|.
\end{aligned}
$$

(c) Define $g \coloneqq \chi_B - \chi_A$. Then $|g| \le 1$ and hence $|\nu|(E) = |\int_E g\, d\nu| \le \sup\{|\int_E f\, d\nu| \mid |f| \le 1\}$ because

$$
\begin{aligned}
\int_E g\, d\nu &= \int (\chi_B - \chi_A)\chi_E \, d\nu \\
&= \int \chi_B\chi_E \, d\nu^+ - \int -\chi_A\chi_E \, d\nu^- \\
&= \int \chi_{(B\cap E)} \, d\nu^+ + \int \chi_{(A\cap E)} \, d\nu^- \\
&= \nu^+(B\cap E) + \nu^-(A\cap E) \\
&= \nu^+(A\cap E) + \nu^+(B\cap E) + \nu^-(A\cap E) + \nu^-(B\cap E) \\
&= \nu^+(E) + \nu^-(E) \\
&= |\nu|(E) \ge 0.
\end{aligned}
$$

This completes the proof if $|\nu|(E) = \infty$. Assume that $|\nu|(E) < \infty$ and let f be a measurable function with $|f| \le 1$. Then $f\chi_E \in L^1(|\nu|) = L^1(\nu)$ because $\int_E |f|\, d|\nu| \le \int \chi_E \, d|\nu| = |\nu|(E) < \infty$. Hence, by the previous exercise $|\int_E f\, d\nu| \le \int |f\chi_E|\, d|\nu| \le \int \chi_E \, d|\nu| = |\nu|(E)$. Therefore $\sup\{|\int_E f\, d\nu| \mid |f| \le 1\} \le |\nu|(E)$.

4. Let A and B be disjoint measurable sets covering X such that A is ν^+-null and B is ν^--null. If E is measurable then

$$\lambda(E) \ge \lambda(E\cap B) = \nu(E\cap B) + \mu(E\cap B) \ge \nu(E\cap B) = \nu^+(E\cap B) = \nu^+(E\cap B) + \nu^+(E\cap A) = \nu^+(E).$$

and similarly

$$\mu(E) \ge \mu(E\cap A) = \mu(E\cap A) + \nu^+(E\cap A) = \lambda(E\cap A) + \nu^-(E\cap A) \ge \nu^-(E\cap A) = \nu^-(E\cap A) + \nu^-(E\cap B) = \nu^-(E).$$

5. Clearly $v_1 + v_2$ is a signed measure, and $v_1 + v_2 = v_1^+ - v_1^- + v_2^+ - v_2^- = (v_1^+ + v_2^+) - (v_1^- + v_2^-)$. Since $v_1^+ + v_2^+$ and $v_1^- + v_2^-$ are positive measures, the previous exercise implies that $v_1^+ + v_2^+ \ge (v_1 + v_2)^+$ and $v_1^- + v_2^- \ge (v_1 + v_2)^-$. Therefore $|v_1 + v_2| = (v_1 + v_2)^+ + (v_1 + v_2)^- \le v_1^+ + v_2^+ + v_1^- + v_2^- = |v_1| + |v_2|$.

7. (a) Let A and B be disjoint measurable sets covering X such that A is ν^+-null and B is ν^--null. Then

$$\nu^+(E) = \nu^+(E\cap B) + \nu^+(E\cap A) = \nu^+(E\cap B) = \nu(E\cap B) \le \sup\{\nu(F) \mid F \in \mathcal{M}, F \subseteq E\}.$$

Moreover, if $F \in \mathcal{M}$ and $F \subseteq E$ then

$$\nu(F) = \nu^+(F) - \nu^-(F) \le \nu^+(F) \le \nu^+(E).$$

Therefore $\nu^+(E) \ge \sup\{\nu(F) \mid F \in \mathcal{M}, F \subseteq E\}$, so $\nu^+(E) = \sup\{\nu(F) \mid F \in \mathcal{M}, F \subseteq E\}$. A similar argument shows that $-\nu^-(E) = \nu(E\cap A)$ and hence $\nu^-(E) = -\inf\{\nu(F) \mid F \in \mathcal{M}, F \subseteq E\}$.

(b) Clearly $|\nu|(E) \leq \sup\{\sum_{i=1}^n |\nu(E_i)| \mid n \in \mathbb{N}$ and $(E_i)_{i=1}^n$ is a pairwise disjoint sequence in $\mathcal{M}$ covering $E\}$, as

$$\begin{aligned}|\nu(E \cap A)| + |\nu(E \cap B)| &= |-\nu^-(E \cap A)| + |\nu^+(E \cap B)| \\ &= \nu^-(E \cap A) + \nu^+(E \cap B) \\ &= \nu^+(E \cap A) + \nu^-(E \cap A) + \nu^+(E \cap B) + \nu^-(E \cap B) \\ &= |\nu|(E \cap A) + |\nu|(E \cap B) \\ &= |\nu|(E).\end{aligned}$$

Now let $n \in \mathbb{N}$ and $(E_i)_{i=1}^n$ is a pairwise disjoint sequence in $\mathcal{M}$ covering E. Then

$$\sum_{i=1}^n |\nu(E_i)| \leq \sum_{i=1}^n (|\nu^+(E_i)| + |\nu^-(E_i)|) = \sum_{i=1}^n \nu^+(E_i) + \sum_{i=1}^n \nu^-(E_i) = \nu^+(E) + \nu^-(E) = |\nu|(E).$$

This shows that $|\nu|(E) \geq \sup\{\sum_{i=1}^n |\nu(E_i)| \mid n \in \mathbb{N}$ and $(E_i)_{i=1}^n$ is a pairwise disjoint sequence in $\mathcal{M}$ covering $E\}$, and hence $|\nu|(E) = \sup\{\sum_{i=1}^n |\nu(E_i)| \mid n \in \mathbb{N}$ and $(E_i)_{i=1}^n$ is a pairwise disjoint sequence in $\mathcal{M}$ covering $E\}$.

8. Suppose $\nu \ll \mu$. If $E \in \mathcal{M}$ and $\mu(E) = 0$ then $\nu(F) = \mu(F) = 0$ for all $F \in \mathcal{M}$ with $F \subseteq E$, so E is ν-null. By exercise 2, this implies that $|\nu|(E) = 0$. Therefore $|\nu| \ll \mu$.

   Now suppose that $|\nu| \ll \mu$. If $E \in \mathcal{M}$ and $\mu(E) = 0$, then $\nu^+(E) + \nu^-(E) = 0$ and hence $\nu^+(E) = \nu^-(E) = 0$. This shows that $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

   Finally, suppose that $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. If $E \in \mathcal{M}$ and $\mu(E) = 0$, then $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - 0 = 0$, and hence $\nu \ll \mu$.

9. Suppose that $\nu_n \perp \mu$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ there exists a measurable set E_n such that E_n is ν_n-null and E_n^c is μ-null. Define $E := \cap_{n=1}^\infty E_n$ and note that $E^c = \cup_{n=1}^\infty E_n^c$; in particular E^c is μ-null. It is also clear that E is a null set with respect to $\sum_{n=1}^\infty \nu_n$. Therefore $\sum_{n=1}^\infty \nu_n \perp \mu$. The second part is trivial.

11. (a) If $f \in L^1(\mu)$ then $\{f\}$ is uniformly integrable, because the finite signed measure $E \mapsto \int_E f\,d\mu$ is absolutely continuous with respect to μ. Hence, if $\{f_i\}_{i\in I}$ is a finite subcollection of $L^1(\mu)$ and $\varepsilon \in (0, \infty)$, for each $i \in I$ there exists $\delta_i \in (0, \infty)$ such that $|\int_E f_i\,d\mu| < \varepsilon$ for all $E \in \mathcal{M}$ with $\mu(E) < \delta_i$. Set $\delta := \min\{\delta_i\}_{i\in I}$, so that $|\int_E f_i\,d\mu| < \varepsilon$ for all $i \in I$ and $E \in \mathcal{M}$ with $\mu(E) < \delta$. This shows that $\{f_i\}_{i\in I}$ is uniformly integrable.

    (b) Let $\varepsilon \in (0, \infty)$, and choose $N \in \mathbb{N}$ such that $\int |f_n - f|\,d\mu < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$ with $n \geq N$. Define $f_0 := f$ and $I := \{n\}_{n=0}^N$. Since $\{f_n\}_{n\in I}$ is uniformly integrable, there exists $\delta \in (0, \infty)$ such that $|\int_E f_n\,d\mu| < \frac{\varepsilon}{2}$ for all $n \in I$ and $E \in \mathcal{M}$ with $\mu(E) < \delta$. If $n \in \mathbb{N} \setminus I$ and $E \in \mathcal{M}$ then

$$\left|\int_E f_n\,d\mu\right| = \left|\int_E (f_n - f)\,d\mu + \int_E f\,d\mu\right| \leq \left|\int_E (f_n - f)\,d\mu\right| + \left|\int_E f\,d\mu\right| \leq \int_E |f_n - f|\,d\mu + \left|\int_E f_0\,d\mu\right|$$

    Since $\int_E |f_n - f|\,d\mu \leq \int |f_n - f|\,d\mu$, this implies that $|\int_E f_n\,d\mu| < \varepsilon$ for all $n \in \mathbb{N}$ and $E \in \mathcal{M}$ with $\mu(E) < \delta$. Therefore $\{f_n\}_{n=1}^\infty$ is uniformly integrable.

13. (a) Let $E \in \mathcal{M}$ and suppose that $\mu(E) = 0$. Then $E = \varnothing$ so $m(E) = 0$. Therefore $m \ll \mu$. Suppose there exists an extended μ-integrable function f such that $dm = f\,d\mu$. If $x \in X$ then $f(x) = \int_{\{x\}} f\,d\mu = m(\{x\}) = 0$, so $f = 0$. Therefore $m(X) = \int_X f\,d\mu = 0$, which is a contradiction because $m(X) = 1$.

(b) Suppose that μ has a Lebesgue decomposition $\lambda+\rho$ with respect to m, with $\lambda \perp m$ and $\rho \ll m$. Then $\rho(\{x\}) = 0$ and hence $\lambda(\{x\}) = \mu(\{x\}) = 1$ for all $x \in X$. There exist disjoint measurable sets A and B covering X such that A is λ-null and $m(B) = 0$. If $x \in A$ then $1 = \lambda(\{x\}) = 0$ so $A = \varnothing$. Therefore $m(B) = m(X) = 1$, which contradicts $m(B) = 0$.

16. For each $n \in \mathbb{N}$ let $E_n \coloneqq \{x \in X \mid f(x) < -n^{-1}\}$, so that

$$-n^{-1}\lambda(E_n) = \int_{E_n} -n^{-1}\, d\lambda \geq \int_{E_n} f\, d\lambda = \nu(E_n) \geq 0$$

and hence $\mu(E_n) \leq \lambda(E_n) = 0$. It follows that $\mu(\cup_{n=1}^{\infty} E_n) = 0$, so $f \geq 0$ μ-a.e. because $\cup_{n=1}^{\infty} E_n = \{x \in X \mid f(x) < 0\}$. Now set $F \coloneqq \{x \in X \mid f(x) \geq 1\}$. Since ν is σ-finite, there is a sequence $(F_n)_{n=1}^{\infty}$ of subsets of F which cover F such that $\nu(F_n) < \infty$ for all $n \in \mathbb{N}$. Since

$$\nu(F_n) = \int_{F_n} f\, d\lambda \geq \int_{F_n} 1\, d\lambda = \lambda(F_n) = \mu(F_n) + \nu(F_n)$$

it is clear that $\mu(F_n) = 0$ for all $n \in \mathbb{N}$, in which case $\mu(F) = 0$ and hence $f < 1$ μ-a.e. Without loss of generality $f, 1-f \in L^+$, so for each $E \in \mathcal{M}$

$$\int_E (1-f)\, d\lambda + \nu(E) = \int_E 1\, d\lambda = \lambda(E) = \mu(E) + \nu(E).$$

In particular $\int_E (1-f)\, d\lambda = \mu(E)$ for all $E \in \mathcal{M}$ with $\nu(E) < \infty$. This result extends to all $E \in \mathcal{M}$ because ν is σ-finite, using the monotone convergence theorem and the additivity of μ. Therefore $\frac{d\mu}{d\lambda} = (1-f)$. Since $\mu \ll \lambda$ (in fact $\mu \leq \lambda$) and $\lambda \ll \mu$ (as $\nu \ll \mu$), it follows that $\frac{d\lambda}{d\mu} = \frac{1}{1-f}$. This implies that

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu} = \frac{f}{1-f}.$$

17. Define a measure λ on $\mathcal{M}$ by $\lambda(E) \coloneqq \int_E f\, d\mu$. Then λ is finite because $f \in L^1(\mu)$. Define $\rho \coloneqq \lambda|_{\mathcal{N}}$. Clearly $\rho \ll \nu$, so by the Radon-Nikodym theorem there is an extended ν-integrable function g such that $\rho(E) = \int_E g\, d\nu$ for all $E \in \mathcal{N}$. In fact $g \in L^1(\nu)$ since $\rho(X) < \infty$. Moreover g is unique up to equality ν-a.e. by the Radon-Nikodym theorem.

19. There exist positive measures ρ and σ and complex-valued functions $f \in L^1(\rho)$ and $g \in L^1(\sigma)$ such that $d\nu = f\, d\rho$, $d|\nu| = |f|\, d\rho$, $d\mu = g\, d\sigma$ and $d|\mu| = |g|\, d\sigma$. Suppose that $\nu \perp \mu$. For each pair $a, b \in \{r, i\}$ there exist disjoint sets $A_{ab}, B_{ab} \in \mathcal{M}$ such that $A_{ab} \cup B_{ab} = X$, A_{ab} is ν_a-null and B_{ab} is μ_b-null. It follows that $A \coloneqq (A_{rr} \cup A_{ri}) \cap (A_{ir} \cup A_{ii})$ is both ν_r-null and ν_i-null. In particular, if $E \in \mathcal{M}$ and $E \subseteq A$ then $0 = \nu_r(E) = \mathrm{Re}(\int_E f\, d\rho) = \int_E \mathrm{Re}(f)\, d\rho$. By applying this to the sets $\{x \in A \mid \mathrm{Re}(f(x)) > n^{-1}\}$ and $\{x \in A \mid \mathrm{Re}(f(x)) < -n^{-1}\}$ for all $n \in \mathbb{N}$, we can show that $\mathrm{Re}(f)\chi_A = 0$ ρ-a.e., and similarly $\mathrm{Im}(f)\chi_A = 0$ ρ-a.e., giving $|f|\chi_A = 0$ ρ-a.e. and hence A is $|\nu|$-null. Moreover

$$B \coloneqq A^c = (A_{rr} \cup A_{ri})^c \cup (A_{ir} \cup A_{ii})^c = (B_{rr} \cap B_{ri}) \cup (B_{ir} \cap B_{ii})$$

is both μ_r-null and μ_i-null, and a similar argument shows that B is $|\mu|$-null. Therefore $|\nu| \perp |\mu|$.

Conversely, suppose that $|\nu| \perp |\mu|$. There exist disjoint sets $A, B \in \mathcal{M}$ such that $A \cup B = X$, A is $|\nu|$-null and B is $|\mu|$-null. It follows that $|f|\chi_A = 0$ ρ-a.e., so that $\mathrm{Re}(f)\chi_A = 0 = \mathrm{Im}(f)\chi_A$ ρ-a.e. and hence A is both ν_r-null and ν_i-null. Similarly B is both μ_r-null and μ_i-null, which implies that $\nu_a \perp \mu_b$ for all $a, b \in \{r, i\}$. This shows that $\nu \perp \mu$.

Now suppose that $\nu \ll \lambda$, and let $A \in \mathcal{M}$ satisfy $\lambda(A) = 0$. If $E \in \mathcal{M}$ and $E \subseteq A$ then $\lambda(E) = 0$ and hence $\nu_a(E) = 0$ for each $a \in \{r, i\}$. It follows that $\int_E \operatorname{Re}(f)\, d\rho = \int_E \operatorname{Im}(f)\, d\rho = 0$ for all $E \in \mathcal{M}$ with $E \subseteq A$, so $|f|\chi_A = 0$ ρ-a.e. which implies that $|\nu|(A) = 0$. This shows that $|\nu| \ll \lambda$.

Conversely, if $|\nu| \ll \lambda$ then $|\nu(A)| \leq |\nu|(A) = 0$, and hence $\nu_a(A) = 0$, for each $a \in \{r, i\}$ and all $A \in \mathcal{M}$ with $\lambda(A) = 0$. This implies that $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$, so $\nu \ll \lambda$.

21. If $n \in \mathbb{N}$ and $E_1, E_2, \ldots, E_n \in \mathcal{M}$ are disjoint with $E = \cup_{k=1}^n E_k$, then $E_1, E_2, \ldots$ are disjoint, $E = \cup_{k=1}^\infty E_k$ and $\sum_{k=1}^\infty |\nu(E_k)| = \sum_{k=1}^n |\nu(E_k)|$ where $E_{n+k} := \varnothing$ for all $k \in \mathbb{N}$. This implies that $\mu_1(E) \leq \mu_2(E)$. Moreover, if $(E_k)_{k=1}^\infty$ is a sequence of disjoint measurable sets such that $E = \cup_{k=1}^\infty E_k$, then

$$f := \sum_{k=1}^\infty \frac{\overline{\nu(E_k)}}{|\nu(E_k)|}\chi_{E_k}$$

is well-defined and satisfies $|f| \leq 1$ (by disjointness). If $a \in \{r, i\}$ and $b \in \{+, -\}$ then $1 \in L^1(\nu_a^b)$ so

$$\int f\, d\nu_a^b = \lim_{n\to\infty} \int \sum_{k=1}^n \frac{\overline{\nu(E_k)}}{|\nu(E_k)|}\chi_{E_k}\, d\nu_a^b = \lim_{n\to\infty} \sum_{k=1}^n \frac{\overline{\nu(E_k)}}{|\nu(E_k)|}\nu_a^b(E_k) = \sum_{k=1}^\infty \frac{\overline{\nu(E_k)}}{|\nu(E_k)|}\nu_a^b(E_k)$$

by the dominated convergence theorem. It follows that

$$\begin{aligned}
\int f\, d\nu &= \int f\, d\nu_r + i\int f\, d\nu_i \\
&= \int f\, d\nu_r^+ - \int f\, d\nu_r^- + i\int f\, d\nu_i^+ - i\int f\, d\nu_i^- \\
&= \sum_{k=1}^\infty \frac{\overline{\nu(E_k)}}{|\nu(E_k)|}(\nu_r^+(E_k) - \nu_r^-(E_k) + i\nu_i^+(E_k) - i\nu_i^-(E_k)) \\
&= \sum_{k=1}^\infty \frac{\overline{\nu(E_k)}}{|\nu(E_k)|}(\nu_r(E_k) + i\nu_i(E_k)) \\
&= \sum_{k=1}^\infty \frac{\overline{\nu(E_k)}}{|\nu(E_k)|}\nu(E_k) \\
&= \sum_{k=1}^\infty \frac{|\nu(E_k)|^2}{|\nu(E_k)|} \\
&= \sum_{k=1}^\infty |\nu(E_k)|.
\end{aligned}$$

This implies that $|\int_E f\, d\nu| = \sum_{k=1}^\infty |\nu(E_k)|$, so $\mu_2(E) \leq \mu_3(E)$. If f is a complex-valued measurable function on X such that $|f| \leq 1$ then $|\int_E f\, d\nu| \leq \int_E |f|\, d|\nu| \leq \int_E 1\, d|\nu| = |\nu|(E)$, so $\mu_3(E) \leq |\nu|(E)$. Conversely, if $f := \frac{d\nu}{d|\nu|}$ (which exists because $\nu \ll |\nu|$) then $|f| = 1$ $|\nu|$-almost everywhere and hence $|\overline{f}| \leq 1$ everywhere (without loss of generality). By generalising the chain rule to complex measures, it follows that

$$\mu_3(E) \geq \left|\int_E \overline{f}\, d\nu\right| = \left|\int_E \overline{f}\frac{d\nu}{d|\nu|}\, d|\nu|\right| = \left|\int_E \overline{f}f\, d|\nu|\right| = \left|\int_E |f|^2\, d|\nu|\right| = \left|\int_E 1\, d|\nu|\right| = ||\nu|(E)| = |\nu|(E).$$

This shows that $\mu_3(E) = |\nu|(E)$, so it remains to show that $\mu_3(E) \leq \mu_1(E)$. If f is a complex-valued measurable function on X such that $|f| \leq 1$, there exists a sequence $(\phi_k)_{k=1}^\infty$ of simple functions which converges pointwise to f

such that $(|\phi_k|)_{k=1}^\infty$ is increasing to $|f|$. For each $k \in \mathbb{N}$ let

$$\phi_k = \sum_{j=1}^{n_k} c_{kj}\chi_{E_{kj}}$$

be the standard representation of ϕ_k. By the dominated convergence theorem, if $a \in \{r, i\}$ and $b \in \{+, -\}$ then

$$\int_E f\, d\nu_a^b = \lim_{k\to\infty} \int_E \phi_k\, d\nu_a^b = \lim_{k\to\infty} \sum_{j=1}^{n_k} c_{kj}\nu_a^b(E_{kj} \cap E)$$

and hence

$$\begin{aligned}
\int_E f\, d\nu &= \int_E f\, d\nu_r + i\int_E f\, d\nu_i \\
&= \int_E f\, d\nu_r^+ - \int_E f\, d\nu_r^- + i\int_E f\, d\nu_i^+ - i\int_E f\, d\nu_i^- \\
&= \lim_{k\to\infty}\left(\sum_{j=1}^{n_k} c_{kj}\nu_r^+(E_{kj}\cap E) - \sum_{j=1}^{n_k} c_{kj}\nu_r^-(E_{kj}\cap E) + i\sum_{j=1}^{n_k} c_{kj}\nu_i^+(E_{kj}\cap E) - i\sum_{j=1}^{n_k} c_{kj}\nu_i^-(E_{kj}\cap E)\right) \\
&= \lim_{k\to\infty}\sum_{j=1}^{n_k} c_{kj}(\nu_r^+(E_{kj}\cap E) - \nu_r^-(E_{kj}\cap E) + i\nu_i^+(E_{kj}\cap E) - i\nu_i^-(E_{kj}\cap E)) \\
&= \lim_{k\to\infty}\sum_{j=1}^{n_k} c_{kj}(\nu_r(E_{kj}\cap E) + i\nu_i(E_{kj}\cap E)) \\
&= \lim_{k\to\infty}\sum_{j=1}^{n_k} c_{kj}\nu(E_{kj}\cap E).
\end{aligned}$$

Since $|\phi_k| \le 1$ and $E_{k1}, E_{k2}, \ldots, E_{kn_k}$ are disjoint for each $k \in \mathbb{N}$, it follows that

$$\left|\int_E f\, d\nu\right| = \lim_{k\to\infty}\left|\sum_{j=1}^{n_k} c_{kj}\nu(E_{kj}\cap E)\right| \le \lim_{k\to\infty}\sum_{j=1}^{n_k} |c_{kj}||\nu(E_{kj}\cap E)| \le \lim_{k\to\infty}\sum_{j=1}^{n_k} |\nu(E_{kj}\cap E)| \le \mu_1(E).$$

Therefore $\mu_3(E) \le \mu_1(E)$, as required.

22. Assuming $\|f\|_1 > 0$, there exists $R \in (0, \infty)$ such that $\int_{B_R(0)} |f|\, dm > 0$ (otherwise the monotone convergence theorem implies that $\int |f|\, dm = \lim_{N\to\infty} \int_{B_N(0)} |f|\, dm = 0$). If $x \in \mathbb{R}^n \setminus B_R[0]$, then $B_R(0) \subseteq B_{2|x|}(x)$ so

$$Hf(x) \ge A_{2|x|}|f|(x) = \frac{1}{m(B_{2|x|}(x))}\int_{B_{2|x|}(x)} |f|\, dm \ge \frac{1}{m(B_{2|x|}(0))}\int_{B_R(0)} |f|\, dm = \frac{1}{|x|^n m(B_2(0))}\int_{B_R(0)} |f|\, dm.$$

Note that $C := \frac{1}{m(B_2(0))}\int_{B_R(0)} |f|\, dm$ is positive and independent of x. Now, if $\alpha \in (0, \frac{C}{2R^n})$ and $x \in \mathbb{R}^n$ such that $R < |x| < (\frac{C}{\alpha})^{1/n}$ then $Hf(x) \ge C|x|^{-n} > C\frac{\alpha}{C} = \alpha$ and hence $B_{(\frac{C}{\alpha})^{1/n}}(0) \setminus B_R[0] \subseteq \{x \in \mathbb{R}^n \mid Hf(x) > \alpha\}$. Thus

$$m(\{x \in \mathbb{R}^n \mid Hf(x) > \alpha\}) \ge m\left(B_{(\frac{C}{\alpha})^{1/n}}(0)\right) - m(B_R[0]) = \left(\frac{C}{\alpha} - R^n\right) m(B_1(0)) > \frac{Cm(B_1(0))}{2\alpha}.$$

23. Let $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, so that $B_r(x)$ is a ball with $x \in B_r(x)$. Then $\frac{1}{m(B_r(x))}\int_{B_r(x)} |f|\, dm \le H^*f(x)$ and hence $Hf(x) \le H^*f(x)$. Conversely, let $y \in \mathbb{R}^n$ and $r \in (0, \infty)$ such that $x \in B_r(y)$. Clearly $B_r(y) \subseteq B_{2r}(x)$, so

$$\frac{1}{m(B_r(y))}\int_{B_r(y)} |f|\, dm \le \frac{1}{m(B_r(y))}\int_{B_{2r}(x)} |f|\, dm = \frac{2^n}{m(B_{2r}(x))}\int_{B_{2r}(x)} |f|\, dm \le 2^n Hf(x)$$

and hence $H^*f(x) \le 2^n Hf(x)$.

24. Let $\varepsilon \in (0,\infty)$ and take $r \in (0,\infty)$ such that $|f(y)-f(x)| < \varepsilon$ for all $y \in B_r(x)$. Then

$$\left|\frac{1}{m(B_r(x))}\int_{B_r(x)}|f(y)-f(x)|\,dy\right| = \frac{1}{m(B_r(x))}\int_{B_r(x)}|f(y)-f(x)|\,dy < \frac{1}{m(B_r(x))}\int_{B_r(x)}\varepsilon\,dy = \frac{\varepsilon m(B_r(x))}{m(B_r(x))} = \varepsilon.$$

This shows that $\lim_{r\downarrow 0}\frac{1}{m(B_r(x))}\int_{B_r(x)}|f(y)-f(x)|\,dy = 0$. Therefore $x \in L_f$.

25. (a) Define $\mu : \mathcal{B}_{\mathbb{R}^n} \to [0,\infty]$ by $\mu(A) = m(E \cap A)$. Clearly μ is a measure with Lebesgue-Radon-Nikodym representation $d\mu = \chi_E\,dm$. Let $A \in \mathcal{B}_{\mathbb{R}^n}$ and suppose $m(A) < \infty$. Given $\varepsilon \in (0,\infty)$, there exists an open set $U \subseteq \mathbb{R}^n$ containing A such that $m(U) < m(A) + \varepsilon$ and hence $m(U \setminus A) < \varepsilon$. It follows that

$$\mu(U) = m(E \cap U) \leq m(E \cap A) + m(U \setminus A) < \mu(A) + \varepsilon.$$

Given $A \in \mathcal{B}_{\mathbb{R}^n}$ and $\varepsilon \in (0,\infty)$, choose a sequence $(A_k)_{k=1}^\infty$ of Borel sets such that $m(A_k) < \infty$ for all $k \in \mathbb{N}$ and $\cup_{k=1}^\infty A_k = A$. For each $k \in \mathbb{N}$ there exists an open set $U_k \subseteq \mathbb{R}^n$ containing A_k such that $\mu(U_k) < \mu(A_k) + 2^{-k}\varepsilon$. Clearly $A \subseteq \cup_{k=1}^\infty U_k$ and $\mu(\cup_{k=1}^\infty U_k \setminus A) < \varepsilon$. This implies that $\mu(A) = \inf\{\mu(U) \mid U \subseteq \mathbb{R}^n \text{ is open and } A \subseteq U\}$. Therefore μ is regular, so for almost every $x \in \mathbb{R}^n$

$$D_E(x) = \lim_{r\downarrow 0}\frac{m(E \cap B_r(x))}{m(B_r(x))} = \lim_{r\downarrow 0}\frac{\mu(B_r(x))}{m(B_r(x))} = \chi_E(x).$$

In particular $D_E(x) = 1$ for almost all $x \in E$ and $D_E(x) = 0$ for almost all $x \in E^c$.

(b) Given $\alpha \in [0, \frac{1}{4})$ define $E_\alpha := \{(x,y) \in (0,\infty)^2 \mid y \leq x\tan(2\pi\alpha)\}$. Fix $r \in (0,\infty)$ and note that $E_\alpha \cap B_r(0)$ is a sector of the disk $B_r(0)$ between the angles 0 and $2\pi\alpha$. Hence $m(E_\alpha \cap B_r(0)) = \alpha m(B_r(0))$ and $D_{E_\alpha}(0) = \alpha$. For $\alpha \in [\frac{1}{4}, 1)$ just include some of the other quadrants $(0,\infty)\times(-\infty,0)$, $(-\infty,0)\times(0,\infty)$ or $(-\infty,0)^2$ in E_α. Now define $E := \cup_{n=1}^\infty[2^{-n}, 2^{-n}+2^{-n-1}]$ and fix $N \in \mathbb{N}$. Then $E \cap B_{2^{-N}}(0) = \cup_{n=N+1}^\infty[2^{-n}, 2^{-n}+2^{-n-1}]$ so

$$m(E \cap B_{2^{-N}}(0)) = \sum_{n=N+1}^\infty m([2^{-n}, 2^{-n}+2^{-n-1}]) = \sum_{n=N+1}^\infty 2^{-n-1} = 2^{-N-1},$$

while $m(B_{2^{-N}}(0)) = 2 \cdot 2^{-N} = 2^{1-N}$. This implies that

$$\frac{m(E \cap B_{2^{-N}}(0))}{m(B_{2^{-N}}(0))} = \frac{2^{-N-1}}{2^{1-N}} = 2^{-2} = \frac{1}{4}.$$

On the other hand $E \cap B_{2^{-N}+2^{-N-1}}(0) = (\cup_{n=N+1}^\infty[2^{-n}, 2^{-n}+2^{-n-1}]) \cup [2^{-N}, 2^{-N}+2^{-N-1})$, which implies that

$$m(E \cap B_{2^{-N}+2^{-N-1}}(0)) = m([2^{-N}, 2^{-N}+2^{-N-1})) + \sum_{n=N+1}^\infty m([2^{-n}, 2^{-n}+2^{-n-1}]) = \sum_{n=N}^\infty 2^{-n-1} = 2^{-N},$$

but $m(B_{2^{-N}+2^{-N-1}}(0)) = 2\cdot(2^{-N}+2^{-N-1}) = 2^{1-N}+2^{-N} = 3\cdot 2^{-N}$ and hence

$$\frac{m(E \cap B_{2^{-N}+2^{-N-1}}(0))}{m(B_{2^{-N}+2^{-N-1}}(0))} = \frac{2^{-N}}{3\cdot 2^{-N}} = \frac{1}{3}.$$

Since 2^{-N} and $2^{-N}+2^{-N-1}$ converge to 0 as $N \to \infty$, it follows that $D_E(0)$ does not exist.

28. (a) If $x \in \mathbb{R}$ then

$$\begin{aligned}
T_F(x) &= \sup\left\{\sum_{k=1}^n |F(x_k) - F(x_{k-1})| \;\middle|\; n \in \mathbb{N} \text{ and } x_0, x_1, \ldots, x_n \in \mathbb{R} \text{ with } x_0 < x_1 < \cdots < x_n = x\right\} \\
&= \sup\left\{\sum_{k=1}^n |\mu_F((-\infty, x_k]) - \mu_F((-\infty, x_{k-1}])| \;\middle|\; n \in \mathbb{N} \text{ and } x_0, x_1, \ldots, x_n \in \mathbb{R} \text{ with } x_0 < x_1 < \cdots < x_n = x\right\} \\
&= \sup\left\{\sum_{k=1}^n |\mu_F((x_{k-1}, x_k])| \;\middle|\; n \in \mathbb{N} \text{ and } x_0, x_1, \ldots, x_n \in \mathbb{R} \text{ with } x_0 < x_1 < \cdots < x_n = x\right\} \\
&\leq \sup\left\{\sum_{k=1}^n |\mu_F(E_k)| \;\middle|\; n \in \mathbb{N} \text{ and } E_1, E_2, \ldots, E_n \in \mathcal{B}_\mathbb{R} \text{ are disjoint with } \cup_{k=1}^n E_k \subseteq (-\infty, x]\right\} \\
&\leq \sup\left\{\sum_{k=1}^n |\mu_F(E_k)| \;\middle|\; n \in \mathbb{N} \text{ and } E_1, E_2, \ldots, E_n \in \mathcal{B}_\mathbb{R} \text{ are disjoint with } \cup_{k=1}^n E_k = (-\infty, x]\right\} \\
&= |\mu_F|((-\infty, x]) \\
&= G(x),
\end{aligned}$$

where the penultimate equality follows from exercise 21.

(b) If $a, b \in \mathbb{R}$ and $a < b$ then

$$T_F(b) - T_F(a) = \sup\left\{\sum_{k=1}^n |F(x_k) - F(x_{k-1})| \;\middle|\; n \in \mathbb{N} \text{ and } x_0, x_1, \ldots, x_n \in \mathbb{R} \text{ with } a = x_0 < x_1 < \cdots < x_n = b\right\}$$

and hence

$$|\mu_F((a, b])| = |\mu_F((-\infty, b]) - \mu_F((-\infty, a])| = |F(b) - F(a)| \leq T_F(b) - T_F(a) = \mu_{T_F}((a, b]).$$

The set $\mathcal{A}$ of all finite disjoint unions of left-open, right-closed intervals is an algebra of subsets of $\mathbb{R}$. If $x \in \mathbb{R}$

$$|\mu_F((-\infty, x])| = \left|\sum_{k=1}^n \mu_F((x-k, x+1-k])\right| \leq \sum_{k=1}^n |\mu_F((x-k, x+1-k])| \leq \sum_{k=1}^n \mu_{T_F}((x-k, x+1-k]) = \mu_{T_F}((-\infty, x])$$

and similarly $|\mu_F((x, \infty))| \leq \mu_{T_F}((x, \infty))$. By the same argument, it follows that $|\mu_F(E)| \leq \mu_{T_F}(E)$ for all $E \in \mathcal{A}$ (because $|\mu_F(\varnothing)| = \mu_{T_F}(\varnothing)$). Define $\mathcal{C} := \{E \in \mathcal{B}_\mathbb{R} \mid |\mu_F(E)| \leq \mu_{T_F}(E)\}$. If $(E_k)_{k=1}^\infty$ is an increasing sequence in $\mathcal{C}$, then $|\mu_F(\cup_{k=1}^\infty E_k)| = |\lim_{k\to\infty} \mu_F(E_k)| = \lim_{k\to\infty} |\mu_F(E_k)| \leq \lim_{k\to\infty} \mu_{T_F}(E_k) = \mu_{T_F}(\cup_{k=1}^\infty E_k)$ and hence $\cup_{k=1}^\infty E_k \in \mathcal{C}$. Similarly $\mathcal{C}$ is closed under countable decreasing intersections (because μ_{T_F} and the real and imaginary parts of μ_F are finite). Therefore $\mathcal{C}$ contains the monotone class generated by $\mathcal{A}$, which is $\mathcal{B}_\mathbb{R}$ by the monotone class lemma and the fact that $\mathcal{B}_\mathbb{R}$ is generated by $\mathcal{A}$. Thus $|\mu_F(E)| \leq \mu_{T_F}(E)$ for all $E \in \mathcal{B}_\mathbb{R}$.

(c) If $E \in \mathcal{B}_\mathbb{R}$ then, by exercise 21,

$$\begin{aligned}
|\mu_F|(E) &= \sup\left\{\sum_{k=1}^\infty |\mu_F(E_k)| \;\middle|\; (E_k)_{k=1}^\infty \text{ is a sequence of disjoint Borel sets such that } E = \cup_{k=1}^\infty E_k\right\} \\
&\leq \sup\left\{\sum_{k=1}^\infty \mu_{T_F}(E_k) \;\middle|\; (E_k)_{k=1}^\infty \text{ is a sequence of disjoint Borel sets such that } E = \cup_{k=1}^\infty E_k\right\} \\
&= \sup\{\mu_{T_F}(E)\}
\end{aligned}$$

$$= \mu_{T_F}(E).$$

In particular, if $x \in \mathbb{R}$ then $G(x) = |\mu_F|((-\infty, x]) \leq \mu_{T_F}((-\infty, x]) = T_F(x)$. Therefore $G = T_F$ by part (a). Since $|\mu_F|$ and μ_{T_F} are finite, it is straightforward to show that $\mathcal{M} := \{E \in \mathcal{B}_\mathbb{R} \mid |\mu_F|(E) = \mu_{T_F}(E)\}$ is a σ-algebra which contains $(-\infty, x]$ for all $x \in \mathbb{R}$. Therefore $\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}$ and hence $|\mu_F|(E) = \mu_{T_F}(E)$ for all $E \in \mathcal{B}_\mathbb{R}$.

29. The total variation of μ_F as a complex measure is the same as the total variation of μ_F as a signed measure, so

$$T_F(x) + F(x) = \mu_{T_F}((-\infty, x]) + \mu_F((-\infty, x]) = |\mu_F|((-\infty, x]) + \mu_F((-\infty, x]) = 2\mu_F^+((-\infty, x])$$

and hence $\mu_P((-\infty, x]) = P(x) = \frac{1}{2}(T_F(x) + F(x)) = \mu_F^+((-\infty, x])$ for all $x \in \mathbb{R}$. Therefore $\mu_P = \mu_F^+$, because $\{E \in \mathcal{B}_\mathbb{R} \mid \mu_P(E) = \mu_F^+(E)\}$ is a σ-algebra containing a set which generates $\mathcal{B}_\mathbb{R}$. Similarly $\mu_N = \mu_F^-$.

30. Let $q : \mathbb{N} \to \mathbb{Q}$ be a surjection and define $f := \sum_{k=1}^\infty 2^{-k}\chi_{(q(k),\infty)}$. Given $x \in \mathbb{R}$ it is clear that

$$f(x) = \sum_{\substack{k\in\mathbb{N} \\ q(k)<x}} 2^{-k}$$

and hence $f(x) \leq f(y)$ for all $y \in (x, \infty)$. Let $\varepsilon \in (0, \infty)$ and choose $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. There exists $\delta \in (0, \infty)$ such that $(x - \delta, x) \cup (x, x + \delta)$ does not contain $q(1), q(2), \ldots, q(N)$. If $y \in (x - \delta, x)$ then

$$f(x) \geq f(y) = \sum_{\substack{k\in\mathbb{N} \\ q(k)<y}} 2^{-k} = f(x) - \sum_{\substack{k\in\mathbb{N} \\ y\leq q(k)<x}} 2^{-k} \geq f(x) - \sum_{k=N+1}^{\infty} 2^{-k} > f(x) - \varepsilon,$$

which shows that $\lim_{y\uparrow x} f(y) = f(x)$. On the other hand, if $y \in (x, x + \delta)$ then

$$f(x) \leq f(y) = \sum_{\substack{k\in\mathbb{N} \\ q(k)<y}} 2^{-k} = f(x) + \sum_{\substack{k\in\mathbb{N} \\ x\leq q(k)<y}} 2^{-k},$$

which implies that $\lim_{y\downarrow x} f(y) = f(x)$ provided that $x \notin \mathbb{Q}$. However, if $x = q(n)$ for some $n \in \mathbb{N}$ then

$$f(x) + 2^{-n} \leq f(y) = f(x) + 2^{-n} + \sum_{\substack{k\in\mathbb{N} \\ x<q(k)<y}} 2^{-k} < f(x) + 2^{-n} + \varepsilon$$

and hence $\lim_{y\downarrow x} f(y) = f(x) + 2^{-n}$. This shows that $\mathbb{Q}$ is the set of discontinuities of f.

31. (a) By the usual differentiation rules F and G are differentiable on $\mathbb{R} \setminus \{0\}$. Moreover,

$$F'(0) = \lim_{h\to 0} \frac{F(h) - F(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(h^{-1})}{h} = \lim_{h\to 0} h \sin(h^{-1}) = 0$$

by the squeeze theorem. Similarly $G'(0) = 0$.

(b) If $x \in \mathbb{R} \setminus \{0\}$ then $F'(x) = 2x \sin(x^{-1}) - \cos(x^{-1})$ and hence $|F'(x)| \leq 2|x||\sin(x^{-1})| + |\cos(x^{-1})| \leq 2|x| + 1$. Therefore $F' \leq 3$ on $[-1, 1]$. Hence, by the mean value theorem, if $a, b \in [-1, 1]$ and $a < b$ there exists $c \in (a, b)$ such that $|F(b) - F(a)| = |F'(c)||b - a| \leq 3(b - a)$. It follows that $F \in BV([-1, 1])$ because

$$T_F(1) - T_F(-1) = \sup\left\{\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| \;\middle|\; n \in \mathbb{N} \text{ and } x_0, x_1, \ldots, x_n \in \mathbb{R} \text{ with } -1 = x_0 < x_1 < \cdots < x_n = 1\right\}$$

$$\begin{aligned}
&\leq \sup\left\{\sum_{k=1}^{n} 3(x_k - x_{k-1}) \;\middle|\; n \in \mathbb{N} \text{ and } x_0, x_1, \ldots, x_n \in \mathbb{R} \text{ with } -1 = x_0 < x_1 < \cdots < x_n = 1\right\} \\
&= \sup\{3x_n - 3x_0 \mid n \in \mathbb{N} \text{ and } x_0, x_1, \ldots, x_n \in \mathbb{R} \text{ with } -1 = x_0 < x_1 < \cdots < x_n = 1\} \\
&= \sup\{3(1-(-1))\} \\
&= 6.
\end{aligned}$$

For each $k \in \mathbb{N} \cup \{0\}$ define $x_k \coloneqq (\pi(k+\frac{1}{2}))^{-1/2} \in [0,1]$. If $n \in \mathbb{N}$, then

$$\sum_{k=1}^{n} |G(x_k) - G(x_{k-1})| = \sum_{k=1}^{n} |x_k^2(-1)^k - x_{k-1}^2(-1)^{k-1}| = \sum_{k=1}^{n}(x_k^2 + x_{k-1}^2) \geq \sum_{k=1}^{n} x_k^2 = \sum_{k=1}^{n} \frac{1}{\pi(k+\frac{1}{2})} \geq \sum_{k=1}^{n} \frac{1}{2\pi k}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, this implies that $G \notin BV([-1,1])$.

33. Define an increasing, right continuous function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(x) \coloneqq \begin{cases} \lim_{y \downarrow x} F(y), & x < b \\ F(b), & x \geq b. \end{cases}$$

Then $G(b) - G(a) \leq F(b) - F(a)$ and $G' = F'$ almost everywhere on (a,b). Moreover, there exists a regular measure μ_G on $\mathbb{R}$ such that $\mu_G((a,b]) = G(b) - G(a)$ for all $a, b \in \mathbb{R}$ with $a < b$. Let $d\mu_G = d\lambda + f\,dm$ be the Lebesgue-Radon-Nikodym representation of μ_G (note that $\lambda \geq 0$ by the proof of Lebesgue-Radon-Nikodym), so that

$$\lim_{h\downarrow 0} \frac{G(x+h) - G(x)}{h} = \lim_{h \downarrow 0} \frac{\mu_G((x, x+h])}{m((x, x+h])} = f(x)$$

and

$$\lim_{h\uparrow 0} \frac{G(x+h) - G(x)}{h} = \lim_{h\downarrow 0} \frac{G(x-h) - G(x)}{-h} = \lim_{h\downarrow 0} \frac{G(x) - G(x-h)}{h} = \lim_{h\downarrow 0} \frac{\mu_G((x-h, x])}{m((x-h,x])} = f(x)$$

for almost all $x \in \mathbb{R}$. In particular

$$F(b) - F(a) \geq G(b) - G(a) = \mu_G((a,b]) \geq \int_{(a,b]} f\,dm = \int_a^b G'\,dm = \int_a^b F'\,dm.$$

35. Since F and G are continuous on $[a,b]$, there exists $M \in (0,\infty)$ such that $|F(x)| \leq M$ and $|G(x)| \leq M$ for all $x \in [a,b]$. Let $\varepsilon \in (0,\infty)$ and choose $\delta \in (0,\infty)$ such that

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| < \frac{\varepsilon}{2M} \quad \text{and} \quad \sum_{k=1}^{n} |G(b_k) - G(a_k)| < \frac{\varepsilon}{2M} \tag{2}$$

for every finite collection of disjoint intervals $(a_1,b_1), (a_2,b_2), \ldots, (a_n,b_n) \subseteq [a,b]$ with $\sum_{k=1}^{n}(b_k - a_k) < \delta$. It follows that FG is absolutely continuous on $[a,b]$, because (2) implies that

$$\begin{aligned}
\sum_{k=1}^{n} |FG(b_k) - FG(a_k)| &\leq \sum_{k=1}^{n} (|F(b_k)G(b_k) - F(b_k)G(a_k)| + |F(b_k)G(a_k) - F(a_k)G(a_k)|) \\
&= \sum_{k=1}^{n} |F(b_k)||G(b_k) - G(a_k)| + \sum_{k=1}^{n} |F(b_k) - F(a_k)||G(a_k)| \\
&\leq \sum_{k=1}^{n} M|G(b_k) - G(a_k)| + \sum_{k=1}^{n} |F(b_k) - F(a_k)|M
\end{aligned}$$

$$\begin{aligned} &< M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}M \\ &= \varepsilon. \end{aligned}$$

Therefore, by the fundamental theorem of calculus for Lebesgue integrals, FG is differentiable almost everywhere on $[a,b]$, $(FG)' \in L^1([a,b],m)$ and $F(b)G(b) - F(a)G(a) = \int_a^b (FG)'\,dm$. Since F and G are differentiable almost everywhere on $[a,b]$ (again by the fundamental theorem), it is clear that $(FG)' = FG' + GF'$ almost everywhere on $[a,b]$ and hence $F(b)G(b) - F(a)G(a) = \int_a^b (FG' + GF')\,dm$.

37. Suppose F is Lipschitz continuous, and choose $M \in (0,\infty)$ such that $|F(y) - F(x)| \leq M|y - x|$ for all $x, y \in \mathbb{R}$. If $\varepsilon \in (0,\infty)$ and $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)$ is a finite collection of disjoint intervals with $\sum_{k=1}^n (b_k - a_k) < \frac{\varepsilon}{M}$ then

$$\sum_{k=1}^n |F(b_k) - F(a_k)| \leq \sum_{k=1}^n M|b_k - a_k| = M\sum_{k=1}^m (b_k - a_k) < M\frac{\varepsilon}{M} = \varepsilon.$$

Therefore F is absolutely continuous, so it is differentiable almost everywhere. It follows that, for almost every $x \in \mathbb{R}$,

$$|F'(x)| = \left|\lim_{y\to x}\frac{F(y) - F(x)}{y - x}\right| = \lim_{y\to x}\frac{|F(y) - F(x)|}{|y - x|} \leq \lim_{y\to x}\frac{M|y - x|}{|y - x|} = \lim_{y\to x} M = M.$$

Conversely, suppose F is absolutely continuous and there exists $M \in [0,\infty)$ such that $|F'| \leq M$ almost everywhere. If $x, y \in \mathbb{R}$ and $x \leq y$ then $|F(y) - F(x)| = |\int_x^y F'\,dm| \leq \int_x^y |F'|\,dm \leq \int_x^y M\,dm = M(y - x) = M|y - x|$ Therefore F is Lipschitz continuous with Lipschitz constant M.

39. Set $F_0 \coloneqq F$, and for each $k \in \mathbb{N} \cup \{0\}$ define a non-negative, increasing, right continuous function $G_k : \mathbb{R} \to \mathbb{R}$ by

$$G_k(x) \coloneqq \begin{cases} \lim_{y\downarrow a} F_k(y), & x < a \\ \lim_{y\downarrow x} F_k(y), & x \in [a,b) \\ F_k(b), & x \geq b. \end{cases}$$

Fix $k \in \mathbb{N} \cup \{0\}$, and note that $G_k' = F_k'$ almost everywhere on $[a,b]$. If $x \in [a,b)$, there is a sequence $(x_n)_{n=1}^\infty$ in (x,b) which decreases to x. For each $n \in \mathbb{N}$ it is clear that $\sum_{k=1}^\infty F_k(x_n)\chi_{\{k\}}$ is integrable with respect to the counting measure ν on $\mathbb{N}$; indeed

$$\int \sum_{k=1}^\infty F_k(x_n)\chi_{\{k\}}\,d\nu = \sum_{k=1}^\infty F_k(x_n) = F(x_n) < \infty.$$

Since each F_k is increasing these functions are dominated by $\sum_{k=1}^\infty F_k(x_1)\chi_{\{k\}}$, and by definition

$$\lim_{n\to\infty}\sum_{k=1}^\infty F_k(x_n)\chi_{\{k\}} = \sum_{k=1}^\infty G_k(x)\chi_{\{k\}}$$

pointwise. Hence, by the dominated convergence theorem

$$G_0(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty}\int\sum_{k=1}^\infty F_k(x_n)\chi_{\{k\}}\,d\nu = \int\sum_{k=1}^\infty G_k(x)\chi_{\{k\}} = \sum_{k=1}^\infty G_k(x).$$

Moreover, if $x \in (-\infty,a)$ then $G_0(x) = G_0(a) = \sum_{k=1}^\infty G_k(a) = \sum_{k=1}^\infty G_k(x)$ and similarly, if $x \in [b,\infty)$ then $G_0(x) = F_0(b) = \sum_{k=1}^\infty F_k(b) = \sum_{k=1}^\infty G_k(x)$. There exists an outer regular Radon measure μ_k on $\mathbb{R}$ such that

$\mu_k((s,t]) = G_k(t) - G_k(s)$ for all $s,t \in \mathbb{R}$ with $s < t$. Let $d\mu_k = d\lambda_k + f_k\,dm$ be the Lebesgue-Radon-Nikodym representation of μ_k (note that $\lambda_k \geq 0$ by the proof of Lebesgue-Radon-Nikodym), so that

$$\lim_{h\downarrow 0} \frac{G_k(x+h) - G_k(x)}{h} = \lim_{h\downarrow 0} \frac{\mu_k((x,x+h])}{m((x,x+h])} = f_k(x)$$

and

$$\lim_{h\uparrow 0} \frac{G_k(x+h) - G_k(x)}{h} = \lim_{h\downarrow 0} \frac{G_k(x-h) - G_k(x)}{-h} = \lim_{h\downarrow 0} \frac{G_k(x) - G_k(x-h)}{h} = \lim_{h\downarrow 0} \frac{\mu_k((x-h,x])}{m((x-h,x])} = f_k(x)$$

for almost all $x \in \mathbb{R}$, and hence $G_k' = f_k$ almost everywhere. By definition $F_k' \geq 0$ almost everywhere. Note that

$$\left(\sum_{k=1}^\infty \mu_k\right)((s,t]) = \sum_{k=1}^\infty \mu_k((s,t]) = \sum_{k=1}^\infty (G_k(s) - G_k(t)) = G_0(s) - G_0(t) = \mu_0((s,t])$$

for all $s,t \in \mathbb{R}$ with $s < t$. This implies that $\sum_{k=1}^\infty \mu_k = \mu_0$, because both measures are outer regular and every open subset of $\mathbb{R}$ is a disjoint union of countably many left-open, right-closed intervals. For each $k \in \mathbb{N}$ choose $A_k \in \mathcal{B}_\mathbb{R}$ such that $\lambda_k(A_k) = 0$ and $m(A_k^c) = 0$. Then $\cap_{k=1}^\infty A_k$ is null with respect to the measure $\sum_{k=1}^\infty \lambda_k$, while its complement $\cup_{k=1}^\infty A_k^c$ is m-null. Moreover, it is clear that the Borel measure ρ defined by $\rho(E) := \int_E \sum_{k=1}^\infty f_k\,dm$ is absolutely continuous with respect to m. Therefore $\lambda_0 = \sum_{k=1}^\infty \lambda_k$ and $f_0 = \sum_{k=1}^\infty f_k$ almost everywhere, by the Lebesgue-Radon-Nikodym theorem. In particular, $F' = f_0 = \sum_{k=1}^\infty f_k = \sum_{k=1}^\infty F_k'$ almost everywhere on $[a,b]$.

40. By construction F_n is continuous and $F_n(\mathbb{R}) = [0,1]$ for all $n \in \mathbb{N}$. Therefore, if $n \in \mathbb{N}$ then

$$\left|G - \sum_{k=1}^n 2^{-k}F_k\right| = \sum_{k=n+1}^\infty 2^{-k}F_k \leq \sum_{k=n+1}^\infty 2^{-k} = 2^{-n}$$

and hence G is the uniform limit of a sequence of continuous functions. This implies that G is continuous. If $x,y \in \mathbb{R}$ and $x < y$ there exists $n \in \mathbb{N}$ such that $[a_n,b_n] \subseteq (x,y)$, so that $F_n(x) = 0$ while $F_n(y) = 1$ and hence

$$G(x) = F_n(x) + \sum_{\substack{k\in\mathbb{N}\\k\neq n}} 2^{-k}F_k(x) < F_n(y) + \sum_{\substack{k\in\mathbb{N}\\k\neq n}} 2^{-k}F_k(x) \leq F_n(y) + \sum_{\substack{k\in\mathbb{N}\\k\neq n}} 2^{-k}F_k(y) = G(y).$$

The complement of the Cantor set C is open, so if $x \in (0,1)\setminus C$ then F is constant on a neighbourhood of x, implying that $F'(x) = 0$. Since $m(C) = 0$ and F is constant on $(-\infty,0)$ and $(1,\infty)$, this shows that $F' = 0$ almost everywhere. If $n \in \mathbb{N}$, the preimages of $(0,1)$ and $(0,1)\setminus C$ under the map $x \mapsto \frac{x-a_n}{b_n-a_n}$ have measures $b_n - a_n$ and $\sum_{k=1}^\infty 2^{k-1}3^{-k}(b_n-a_n) = b_n - a_n$, which implies that $F_n' = 0$ almost everywhere by the chain rule. Therefore, by the previous exercise $G' = \sum_{k=1}^\infty 2^{-k}F_k' = 0$ almost everywhere.

41. (a) If $x,y \in [0,1]$ and $x < y$ then $F(y) = m([0,y]\cap A) = m([0,x]\cap A) + m((x,y]\cap A) > F(x)$, so F is strictly increasing on $[0,1]$. Let $\varepsilon \in (0,\infty)$. If $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)$ is a finite collection of disjoint intervals such that $\sum_{k=1}^n (b_k - a_k) < \varepsilon$ then

$$\sum_{k=1}^n |F(b_k) - F(a_k)| = \sum_{k=1}^n |m([0,b_k]\cap A) - m([0,a_k]\cap A)| = \sum_{k=1}^n m((a_k,b_k]\cap A) \leq \sum_{k=1}^n m((a_k,b_k]) < \varepsilon,$$

so F is absolutely continuous on $\mathbb{R}$. Hence there exists an outer regular Radon measure μ_F on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a) = \int_a^b F'\,dm$ for all $a,b \in \mathbb{R}$ with $a < b$. Moreover

$$F(b) - F(a) = m([0,b]\cap A) - m([0,a]\cap A) = m((a,b]\cap A) = \int_a^b \chi_A\,dm$$

for all $a, b \in \mathbb{R}$ with $a < b$. This implies that $d\mu_F = F'\,dm = \chi_A\,dm$, since the subset of $\mathcal{B}_\mathbb{R}$ on which these (finite) measures agree is a σ-algebra containing a set which generates $\mathcal{B}_\mathbb{R}$. By the Lebesgue-Radon-Nikodym theorem $F' = \chi_A$ almost everywhere, so $F' = 0$ on $[0,1] \setminus A$, which has positive measure because $m([0,1] \cap A) < m([0,1])$.

(b) Let $\varepsilon \in (0, \infty)$. If $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ is a finite collection of disjoint intervals such that $\sum_{k=1}^n (b_k - a_k) < \varepsilon$

$$\begin{aligned}\sum_{k=1}^n |G(b_k) - G(a_k)| &= \sum_{k=1}^n |m([0,b_k] \cap A) - m([0,b_k] \setminus A) - m([0,a_k] \cap A) + m([0,a_k] \setminus A)| \\ &= \sum_{k=1}^n |m((a_k,b_k] \cap A) - m((a_k,b_k] \setminus A)| \\ &\leq \sum_{k=1}^n (m((a_k,b_k] \cap A) + m((a_k,b_k] \setminus A)) \\ &= \sum_{k=1}^n m((a_k,b_k]) \\ &< \varepsilon,\end{aligned}$$

so G is absolutely continuous on $\mathbb{R}$. In particular G is differentiable on some $E \subseteq \mathbb{R}$ with $m(E^c) = 0$. Moreover

$$\begin{aligned}\int_a^b G'\,dm &= G(b) - G(a) \\ &= m([0,b] \cap A) - m([0,b] \setminus A) - m([0,a] \cap A) + m([0,a] \setminus A) \\ &= m((a,b] \cap A) - m((a,b] \setminus A) \\ &= \int_a^b \chi_A\,dm - \int_a^b \chi_{A^c}\,dm \\ &= \int_a^b (\chi_A - \chi_{A^c})\,dm\end{aligned}$$

for all $a, b \in \mathbb{R}$ such that $a < b$. This implies that $G' = (\chi_A - \chi_{A^c})$ almost everywhere, by the Lebesgue-Radon-Nikodym theorem applied to each of the measures $G'\,dm$ and $(\chi_A - \chi_{A^c})\,dm$ (which are equal because they are finite and agree on a generating set of $\mathcal{B}_\mathbb{R}$). If $a, b \in [0,1]$, $a < b$ and G is monotone on (a,b), then $G' \geq 0$ or $G' \leq 0$ on $(a,b) \cap E$ (because the difference quotient at each point is either non-negative or non-positive). This is a contradiction because $\chi_A - \chi_{A^c}$ takes on the values 1 and -1 on A and A^c respectively, both of which have positive measure and hence $G'(x) = 1$ and $G'(y) = -1$ for some $x, y \in (a,b) \cap E$.

42. (a) Suppose F is convex and let $s, t, s', t' \in (a,b)$ such that $s \leq s' < t'$ and $s < t \leq t'$. Then $\frac{t-s}{t'-s} \in (0,1]$ and hence

$$F(t) = F\left(\frac{t-s}{t'-s}(t'-s) + s\right) = F\left(\frac{t-s}{t'-s}t' + \left(1 - \frac{t-s}{t'-s}\right)s\right) \leq \frac{t-s}{t'-s}F(t') + \left(1 - \frac{t-s}{t'-s}\right)F(s),$$

because if $\frac{t-s}{t'-s} = 1$ then $t = t'$. Moreover $\frac{s'-s}{t'-s} \in [0,1)$, and $\frac{s'-s}{t'-s} = 0$ iff $s' = s$, so

$$F(s') = F\left(\frac{s'-s}{t'-s}(t'-s) + s\right) = F\left(\frac{s'-s}{t'-s}t' + \left(1 - \frac{s'-s}{t'-s}\right)s\right) \leq \frac{s'-s}{t'-s}F(t') + \left(1 - \frac{s'-s}{t'-s}\right)F(s).$$

It follows that

$$F(t) - F(s) \leq \frac{t-s}{t'-s}F(t') - \frac{t-s}{t'-s}F(s)$$

$$\begin{aligned}
&\leq \frac{t-s}{t'-s}F(t') - \frac{t-s}{t'-s}\left(1-\frac{s'-s}{t'-s}\right)^{-1}\left(F(s') - \frac{s'-s}{t'-s}F(t')\right)\\
&= \frac{t-s}{t'-s}F(t') - (t-s)(t'-s-(s'-s))^{-1}\left(F(s') - \frac{s'-s}{t'-s}F(t')\right)\\
&= \frac{t-s}{t'-s}F(t') - \frac{t-s}{t'-s'}\left(F(s') - \frac{s'-s}{t'-s}F(t')\right)\\
&= \left(\frac{t-s}{t'-s} + \frac{t-s}{t'-s'}\frac{s'-s}{t'-s}\right)F(t') - \frac{t-s}{t'-s'}F(s')\\
&= (t-s)\frac{t'-s'+s'-s}{(t'-s)(t'-s')}F(t') - \frac{t-s}{t'-s'}F(s')\\
&= \frac{t-s}{t'-s'}(F(t')-F(s'))
\end{aligned}$$

and hence

$$\frac{F(t)-F(s)}{t-s} \leq \frac{F(t')-F(s')}{t'-s'}. \tag{3}$$

Conversely, suppose that (3) holds for all $s, t, s', t' \in (a,b)$ with $s \leq s' < t'$ and $s < t \leq t'$. Let $x, y \in (a,b)$ and $\lambda \in (0,1)$. Also, set $z := \lambda y + (1-\lambda)x$. If $x < y$, then $x < z < y$ so

$$\frac{F(z)-F(x)}{z-x} \leq \frac{F(y)-F(z)}{y-z},$$

which implies that

$$\frac{F(z)}{z-x} + \frac{F(z)}{y-z} \leq \frac{F(y)}{y-z} + \frac{F(x)}{z-x}$$

and hence

$$\begin{aligned}
F(z) &\leq \left(\frac{1}{z-x} + \frac{1}{y-z}\right)^{-1}\left(\frac{F(y)}{y-z} + \frac{F(x)}{z-x}\right)\\
&= \left(\frac{y-z+z-x}{(z-x)(y-z)}\right)^{-1}\left(\frac{F(y)}{y-z} + \frac{F(x)}{z-x}\right)\\
&= \frac{1}{y-x}((z-x)F(y) + (y-z)F(x))\\
&= \frac{1}{y-x}((\lambda y - \lambda x)F(y) + ((1-\lambda)y - (1-\lambda)x)F(x))\\
&= \lambda F(y) + (1-\lambda)F(x).
\end{aligned}$$

We obtain the same result for the case $x > y$ by swapping x with y and replacing λ with $1-\lambda$. Finally, if $x = y$ then $F(z) = F(x) = \lambda F(x) + (1-\lambda)F(x) = \lambda F(y) + (1-\lambda)F(x)$. Therefore F is convex.

(b) Suppose that F is convex, and let $[t, s'] \subseteq (a,b)$. There exists $s \in (a,t)$ and $t' \in (s', b)$. Let $\varepsilon \in (0,\infty)$ and set

$$M := \max\left\{\frac{F(t')-F(s')}{t'-s'}, \frac{F(s)-F(t)}{t-s}\right\}.$$

If $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n) \subseteq [t,s']$ is a finite collection of disjoint intervals with $\sum_{k=1}^n (b_k - a_k) < \frac{\varepsilon}{M}$ then

$$-M \leq \frac{F(t)-F(s)}{t-s} \leq \frac{F(b_k)-F(a_k)}{b_k-a_k} \leq \frac{F(t')-F(s')}{t'-s'} \leq M$$

for all $k \in \{1, 2, \dots, n\}$ and hence

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| \leq \sum_{k=1}^{n} M|b_k - a_k| = M\sum_{k=1}^{n}(b_k - a_k) < M\frac{\varepsilon}{M} = \varepsilon.$$

Thus F is absolutely continuous on $[t, s']$, and differentiable on some $E \subseteq (a, b)$. If $x_0, y_0 \in E$ and $x_0 < y_0$, then

$$\frac{F(x) - F(x_0)}{x - x_0} \leq \frac{F(y) - F(y_0)}{y - y_0}$$

for all $x \in (x_0, y_0)$ and $y \in (y_0, b)$, so

$$F'(x_0) = \lim_{x \downarrow x_0} \frac{F(x) - F(x_0)}{x - x_0} \leq \frac{F(y) - F(y_0)}{y - y_0}$$

for all $y \in (y_0, b)$ and hence

$$F'(x_0) \leq \lim_{y \downarrow y_0} \frac{F(y) - F(y_0)}{y - y_0} = F'(y_0).$$

This implies that F' is increasing on E.

Conversely, suppose that F is absolutely continuous and F' is increasing on the set $E \subseteq (a, b)$ where F is differentiable. Since F is absolutely continuous, $m((a, b) \setminus E) = 0$. Let $x, y \in (a, b)$, $\lambda \in (0, 1)$ and suppose that $x < y$. Define $z := \lambda y + (1 - \lambda)x$ and $T : [x, y] \to [x, z]$ by $T(t) := \lambda t + (1 - \lambda)x$. Note that $F \circ T$ is absolutely continuous, because if $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \subseteq [x, y]$ is a finite collection of disjoint intervals, then so is $(T(a_1), T(b_1)), (T(a_2), T(b_2)), \dots, (T(a_n), T(b_n)) \subseteq [x, z]$, and

$$\sum_{k=1}^{n}(T(b_k) - T(a_k)) = \sum_{k=1}^{n}(\lambda b_k - \lambda a_k) = \lambda\sum_{k=1}^{n}(b_k - a_k) < \sum_{k=1}^{n}(b_k - a_k).$$

Since $T(t) \leq \lambda t + (1 - \lambda)t = t$ for all $t \in [x, y]$ and F' is increasing on E, it follows that

$$F(T(y)) - F(T(x)) = \int_x^y (F \circ T)'\, dm = \int_x^y F'(T(t))T'(t)\, dt = \lambda\int_x^y F'(T(t))\, dt \leq \lambda\int_x^y F'(t)\, dt = \lambda(F(y) - F(x)).$$

Therefore

$$F(z) = F(T(y)) - F(T(x)) + F(x) \leq \lambda(F(y) - F(x)) + F(x) = \lambda F(y) + (1 - \lambda)F(x),$$

which implies that F is convex by the same argument used in part (a) for the cases $x > y$ and $x = y$.

(c) Fix $t_0 \in (a, b)$ and let $t_1, t_2, t_3, t_4 \in (a, b)$ such that $t_1 \leq t_2 < t_0 < t_3 \leq t_4$. From part (a)

$$\frac{F(t_0) - F(t_1)}{t_0 - t_1} \leq \frac{F(t_0) - F(t_2)}{t_0 - t_2} \leq \frac{F(t_3) - F(t_0)}{t_3 - t_0} \leq \frac{F(t_4) - F(t_0)}{t_4 - t_0},$$

which implies that $\lim_{t \uparrow t_0} \frac{F(t_0) - F(t)}{t_0 - t}$ and $\lim_{t \downarrow t_0} \frac{F(t) - F(t_0)}{t - t_0}$ exist (by monotone convergence) and

$$\frac{F(t_0) - F(t')}{t_0 - t'} \leq \lim_{t \uparrow t_0} \frac{F(t_0) - F(t)}{t_0 - t} \leq \lim_{t \downarrow t_0} \frac{F(t) - F(t_0)}{t - t_0} \leq \frac{F(t'') - F(t_0)}{t'' - t_0}$$

(by taking one limit at a time) for all $t' \in (a, t_0)$ and $t'' \in (t_0, b)$. Set $\beta := \lim_{t \uparrow t_0} \frac{F(t_0) - F(t)}{t_0 - t}$ and let $t \in (a, b)$. If $t > t_0$ then $F(t) - F(t_0) \geq \beta(t - t_0)$, and if $t = t_0$ then $F(t) - F(t_0) = 0 = \beta(t - t_0)$. Otherwise $t < t_0$, in which case $F(t_0) - F(t) \leq \beta(t_0 - t)$ and hence $F(t) - F(t_0) \geq \beta(t - t_0)$.

(d) Since $b - g > 0$ everywhere, $\int (b - g)\, d\mu > 0$ and hence $\int g\, d\mu < \int b\, d\mu = b$. Similarly $a < \int g\, d\mu$, so there exists $\beta \in \mathbb{R}$ such that $F(t) - F(\int g\, d\mu) \geq \beta(t - \int g\, d\mu)$ for all $t \in (a, b)$. In particular

$$\begin{aligned}
\int F \circ g\, d\mu &\geq \int \left(F\left(\int g\, d\mu \right) + \beta \left(g - \int g\, d\mu \right) \right) d\mu \\
&= F\left(\int g\, d\mu \right) \int 1\, d\mu + \beta \int g\, d\mu - \beta \int g\, d\mu \int 1\, d\mu \\
&= F\left(\int g\, d\mu \right) + \beta \int g\, d\mu - \beta \int g\, d\mu \\
&= F\left(\int g\, d\mu \right).
\end{aligned}$$