2. Let $x, y \in X$ and suppose that $x \neq y$. Then $\{x\}^c$ is open in the cofinite topology and contains y but not x. The cofinite topology on X is therefore T_1.

   Since X is infinite it contains two distinct points x and y. Suppose there exist disjoint open sets A and B (in the cofinite topology) such that $x \in A$ and $y \in B$. Then $A \subseteq B^c$, which is finite so A is also finite. This is a contradiction because A^c is finite but $X = A \cup A^c$ is infinite. Hence, the cofinite topology on X is not T_2.

   Suppose that X is countable, and let $x \in X$. Choose a surjection $q : \mathbb{N} \to \{x\}^c$, and for each $n \in \mathbb{N}$ set $A_n := \{q(k)\}_{k=1}^n$ and $B_n := A_n^c$, so that $x \in B_n$ and B_n is open. If U is open and $x \in U$ then $U^c \subseteq \{x\}^c$ is finite and hence $U^c \subseteq A_N$ for some sufficiently large $N \in \mathbb{N}$. This implies that $B_N \subseteq U$, so $\{B_n\}_{n=1}^\infty$ is a neighbourhood base at x. The cofinite topology on X is therefore first countable.

   Conversely, suppose that the cofinite topology on X is first countable. Choose $x \in X$ and a countable neighbourhood base $\{A_n\}_{n=1}^\infty$ at x. If $y \in \{x\}^c$ then $\{y\}^c$ is an open neighbourhood of x, so there exists $n \in \mathbb{N}$ such that $A_n \subseteq \{y\}^c$, and hence $y \in A_n^c$. It follows that $\{x\}^c \subseteq \cup_{n=1}^\infty A_n^c$, which is countable because each A_n^c is finite. Therefore $X = \{x\} \cup \{x\}^c$ is also countable.

3. Let (X, ρ) be a metric space with closed subspaces A and B. Given $x \in X$ and $\varepsilon \in (0, \infty)$, let $y \in B_\varepsilon(x)$. If $a \in A$ then $\rho(x, A) \leq \rho(x, a) \leq \rho(x, y) + \rho(y, a) < \rho(y, a) + \varepsilon$, so $\rho(x, A) - \varepsilon \leq \rho(y, A)$. Similarly $\rho(y, A) - \varepsilon \leq \rho(x, A)$, so $-\varepsilon \leq \rho(y, A) - \rho(x, A) \leq \varepsilon$. This shows that $x \mapsto \rho(x, A)$ is continuous. Similarly $x \mapsto \rho(x, B)$ is continuous.

   It follows that $x \mapsto \rho(x, A) - \rho(x, B)$ is continuous, so the preimages of $(-\infty, 0)$ and $(0, \infty)$ under this map are open. These preimages are $\{x \in X \mid \rho(x, A) < \rho(x, B)\}$ and $\{x \in X \mid \rho(x, A) > \rho(x, B)\}$, which are clearly disjoint. The first set contains A and the second contains B because $\rho(x, A) = 0$ iff $x \in A$ and $\rho(x, B) = 0$ iff $x \in B$. Indeed, if $x \in X$ and $\rho(x, A) = 0$ then every neighbourhood of x meets A, because no $r \in (0, \infty)$ is a lower bound for $\{\rho(x, a)\}_{a \in A}$. Finally, (X, ρ) is T_1 because distinct points are separated by positive distance. Therefore (X, ρ) is normal.

4. Firstly $\varnothing, \mathbb{R} \in \mathcal{T}$ because $\varnothing = \varnothing \cup (\varnothing \cap \mathbb{Q})$ and $\mathbb{R} = \mathbb{R} \cup (\varnothing \cap \mathbb{Q})$. If $\{W_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ and W_α has decomposition $U_\alpha \cup (V_\alpha \cap \mathbb{Q})$ for each $\alpha \in A$, then $\cup_{\alpha \in A} W_\alpha = (\cup_{\alpha \in A} U_\alpha) \cup ((\cup_{\alpha \in A} V_\alpha) \cap \mathbb{Q}) \in \mathcal{T}$. Moreover, if $A = \{1, 2\}$ then

$$\begin{aligned}
W_1 \cap W_2 &= (U_1 \cup (V_1 \cap \mathbb{Q})) \cap (U_2 \cup (V_2 \cap \mathbb{Q})) \\
&= (U_1 \cap U_2) \cup (U_1 \cap (V_2 \cap \mathbb{Q})) \cup ((V_1 \cap \mathbb{Q}) \cap U_2) \cup ((V_1 \cap \mathbb{Q}) \cap (V_2 \cap \mathbb{Q})) \\
&= (U_1 \cap U_2) \cup (((U_1 \cap V_2) \cup (V_1 \cap U_2) \cup (V_1 \cap V_2)) \cap \mathbb{Q}) \\
&\in \mathcal{T}.
\end{aligned}$$

   By induction it follows that $\mathcal{T}$ is closed under finite intersections, so $\mathcal{T}$ is a topology.

   Let $x, y \in \mathbb{R}$ be distinct. There exist disjoint open intervals $U, V \subseteq \mathbb{R}$ such that $x \in U$ and $y \in V$. Since $U = U \cup (\varnothing \cap \mathbb{Q})$ and $V = V \cup (\varnothing \cap \mathbb{Q})$, both these intervals are open in $\mathcal{T}$. Therefore $\mathcal{T}$ is Hausdorff.

   Note that $\mathbb{Q}^c$ is closed in $\mathcal{T}$. Let $W_1, W_2 \in \mathcal{T}$ be neighbourhoods of 0 and $\mathbb{Q}^c$ respectively. Since $0 \in W_1$, there exists $n \in \mathbb{N}$ such that $(-\frac{1}{n}, \frac{1}{n}) \cap \mathbb{Q} \subseteq W_1$. If W_2 has decomposition $U_2 \cup (V_2 \cap \mathbb{Q})$, then $\mathbb{Q}^c \subseteq U_2$ because $(V_2 \cap \mathbb{Q}) \cap \mathbb{Q}^c = \varnothing$. In particular $U_2 \cap (-\frac{1}{n}, \frac{1}{n}) \neq \varnothing$, because it contains, say, $\frac{\sqrt{2}}{2n}$. This implies that $U_2 \cap (-\frac{1}{n}, \frac{1}{n}) \cap \mathbb{Q} \neq \varnothing$, since $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to the usual topology. Therefore $W_2 \cap W_1 \neq \varnothing$, so $\mathcal{T}$ is not regular.

5. Let X be a separable metric space with a countable dense subspace A. Then $\mathcal{B} := \{B_{1/n}(a) \mid a \in A, n \in \mathbb{N}\}$ is countable because $\mathbb{N} \times \mathbb{N}$ is countable. Clearly each member of $\mathcal{B}$ is open in X. Conversely, if $U \subseteq X$ is open and $x \in U$, then $x \in B_{1/n}(x) \subseteq U$ for some $n \in \mathbb{N}$. If $x \in A$ then $B_{1/n}(x) \in \mathcal{B}$. Otherwise $x \in \operatorname{acc}(A)$ because

$X = \overline{A} = A \cup \operatorname{acc}(A)$. It follows that $B_{1/2n}(x)$ contains some point $a \in A$, in which case $x \in B_{1/2n}(a) \in \mathcal{B}$. By the triangle inequality $B_{1/2n}(a) \subseteq B_{1/n}(x) \subseteq U$. This shows that U is the union of a (possibly empty) subcollection of $\mathcal{B}$. Therefore $\mathcal{B}$ is a base for the topology on X, so this topology is second countable.

6. (a) Clearly $\mathbb{R} = \cup_{n=1}^{\infty}(-n, n]$. Let $a, b, c, d \in \mathbb{R}$ and suppose that $a < b$ and $c < d$. Then $(a, b] \cap (c, d] = \varnothing$ or $\max\{a, c\} < \min\{b, d\}$, in which case $(a, b] \cap (c, d] = (\max\{a, c\}, \min\{b, d\}]$. This shows that $\mathcal{E}$ is a base for a topology $\mathcal{T}$ on $\mathbb{R}$. Clearly each member of $\mathcal{E}$ is open in $\mathcal{T}$. If $a, b \in \mathbb{R}$ and $a < b$ then

$$(\cup_{n=1}^{\infty}(a - n, a - n + 1]) \cup (\cup_{n=1}^{\infty}(b + n - 1, b + n])$$

is open in $\mathcal{T}$, so its complement $(a, b]$ is closed in $\mathcal{T}$.

(b) Let $x \in \mathbb{R}$ and note that $\mathcal{N} := \{(x - \frac{1}{n}, x]\}_{n=1}^{\infty}$ is a collection of open neighbourhoods of x. If $U \in \mathcal{T}$ contains x then (since $\mathcal{E}$ is a base for $\mathcal{T}$) there exist $a, b \in \mathbb{R}$ such that $a < b$ and $x \in (a, b] \subseteq U$. It follows that $(x - \frac{1}{n}, x] \subseteq U$ for some $n \in \mathbb{N}$ (with $\frac{1}{n} \leq x - a$). Therefore $\mathcal{N}$ is a countable neighbourhood base at x, so $\mathcal{T}$ is first countable.

Let $\mathcal{B}$ be a base for $\mathcal{T}$, and let $x \in \mathbb{R}$. Then $\mathcal{B}$ contains a neighbourhood base at x, so there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq (x - 1, x]$. In particular $\sup U_x = x$, which implies that $x \mapsto U_x$ is injective. Therefore $\mathcal{B}$ is uncountable. This implies that $\mathcal{T}$ is not second countable.

(c) Let $x \in \mathbb{R}$ and let $U \in \mathcal{T}$ contain x. Then $x \in (a, b] \subseteq U$ for some $a, b \in \mathbb{R}$ with $a < b$. There exists $q \in (a, x) \cap \mathbb{Q}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to the usual topology. Clearly $q \in U \setminus \{x\}$, so $x \in \operatorname{acc}(\mathbb{Q})$ with respect to $\mathcal{T}$. Therefore $\operatorname{acc}(\mathbb{Q}) = \mathbb{R}$, so $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to $\mathcal{T}$. In particular $\mathcal{T}$ is separable.

7. Suppose $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ which converges to x. If U is a neighbourhood of x, there exists $N \in \mathbb{N}$ such that $x_{n_k} \in U$ for all $k \in \mathbb{N}$ with $k \geq N$, hence for infinitely many k. Therefore x is a cluster point of $(x_n)_{n=1}^{\infty}$.

Conversely, suppose that x is a cluster point of $(x_n)_{n=1}^{\infty}$. Since X is first countable, there exists a nested countable neighbourhood base $\{U_n\}_{n=1}^{\infty}$ at x. Set $n_0 := 0$ and, for each $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ inductively so that $n_k > n_{k-1}$ and $x_{n_k} \in U_k$. This is possible because $\{m \in \mathbb{N} \mid x_m \in U_k\}$ is infinite and hence $\{m \in \mathbb{N} \mid x_m \in U_k\} \setminus [1, n_{k-1}] \neq \varnothing$. If U is a neighbourhood of x then $U_N \subseteq U$ for some $N \in \mathbb{N}$. It follows that $x_{n_k} \in U_k \subseteq U_N \subseteq U$ for all $k \in \mathbb{N}$ with $k \geq N$. This shows that the subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ converges to x.

10. (a) Suppose X is connected, and let $A \subseteq X$ be clopen. Then A and A^c are disjoint open sets which cover X. Since X is connected, it follows that $A = \varnothing$ or $A^c = \varnothing$. Therefore $\varnothing$ and X are the only clopen subsets of X.

Conversely, suppose that $\varnothing$ and X are the only clopen subsets of X. If $U, V \subseteq X$ are disjoint open sets which cover X, then $U = V^c$ and hence U is clopen. This implies that $U \in \{\varnothing, X\}$, so $U = \varnothing$ or $V = \varnothing$. Therefore X is not disconnected, i.e. it is connected.

(b) Define $E := \cup_{\alpha \in A} E_\alpha$ and suppose that $U, V \subseteq E$ are non-empty open sets (relative to E) which cover E. Choose $x \in \cap_{\alpha \in A} E_\alpha$, and without loss of generality assume $x \in U$. Also choose $y \in V$, so that $y \in E_\alpha$ for some $\alpha \in A$. Now $x \in U \cap E_\alpha$ and $y \in V \cap E_\alpha$, so $U \cap E_\alpha$ and $V \cap E_\alpha$ are non-empty open sets (relative to E_α) which cover E_α. Since E_α is connected, these sets cannot be disjoint. Therefore $U \cap V \neq \varnothing$, which shows that E is connected.

(c) Let $U, V \subseteq \overline{A}$ be disjoint open sets (relative to $\overline{A}$) which cover $\overline{A}$. Then $U = U' \cap \overline{A}$ and $V = V' \cap \overline{A}$ for some open sets $U', V' \subseteq X$ which cover $\overline{A}$ such that $U' \cap V' \cap \overline{A} = \varnothing$. It follows that $U' \cap A$ and $V' \cap A$ are disjoint open sets (relative to A) which cover A. Since A is connected, $U' \cap A = \varnothing$ without loss of generality. This implies that $U' \cap \operatorname{acc}(A) = \varnothing$, because if $x \in U' \cap \operatorname{acc}(A)$ then $(U' \setminus \{x\}) \cap A \neq \varnothing$, which is impossible. Therefore $U' \cap \overline{A} = \varnothing$, because $\overline{A} = A \cup \operatorname{acc}(A)$, and hence $\overline{A}$ is connected.

(d) Let $x \in X$ and set $\mathcal{C} := \{A \subseteq X \mid A \text{ is connected and } x \in A\}$. Then $C := \cup\mathcal{C}$ is connected by part (b). If $A \subseteq X$ is connected and $C \subseteq A$ then $x \in A$ and hence $A \in \mathcal{C}$. This implies that $A \subseteq C$, so C is maximal. Similarly, if $C' \subseteq X$ is a maximal connected set containing x, then $C' \in \mathcal{C}$ and hence $C' \subseteq C$, so $C' = C$ because C' is maximal. Therefore C is unique. Since $\overline{C}$ is connected (by part (c)) and $C \subseteq \overline{C}$, it follows that $C = \overline{C}$, which means C is closed.

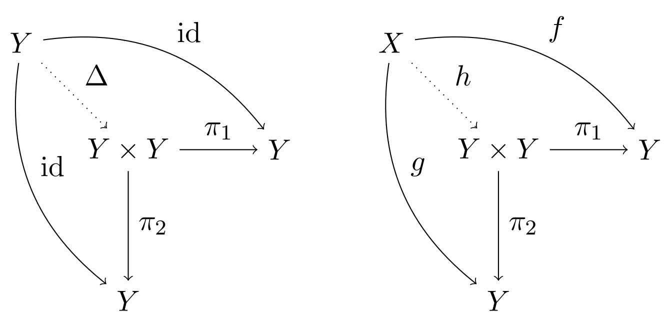
13. Clearly $\overline{U \cap A} \subseteq \overline{U}$. Conversely, note that A is contained in the closed set $U^c \cup \overline{(U \cap A)}$, and hence $U^c \cup \overline{(U \cap A)} = X$. This implies that $U \subseteq \overline{U \cap A}$, so $\overline{U} \subseteq \overline{U \cap A}$. Therefore $\overline{U} = \overline{U \cap A}$.

14. Suppose that f is continuous and let $A \subseteq X$. Then $f^{-1}(\overline{f(A)})$ is closed (as $\overline{f(A)}$ is closed) and $A \subseteq f^{-1}(\overline{f(A)})$ because $f(A) \subseteq \overline{f(A)}$. It follows that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$, which implies that $f(\overline{A}) \subseteq \overline{f(A)}$.

Now suppose that $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. If $B \subseteq Y$, then $f^{-1}(B) \subseteq X$ and hence $f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))}$. Since $f(f^{-1}(B)) \subseteq B$, this implies that $f(\overline{f^{-1}(B)}) \subseteq \overline{B}$, or equivalently $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Finally, suppose that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subseteq Y$. If $C \subseteq Y$ is closed, then $\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C}) = f^{-1}(C)$. This implies that $\overline{f^{-1}(C)} = f^{-1}(C)$, so C is closed and hence f is continuous.

16. (a) Consider the following diagrams:



There exist unique maps $\Delta : Y \to Y \times Y$ and $h : X \to Y \times Y$ which make the resulting diagrams commute, and they are both continuous by Proposition 4.11. Note that $\Delta(Y) = \{(y, y) \mid y \in Y\}$ is closed, because $\Delta(Y)^c$ is open: if $(a, b) \in \Delta(Y)^c$ then $a \neq b$ and there exist disjoint open neighbourhoods U and V of a and b respectively; in particular $(a, b) \in U \times V \subseteq \Delta(Y)^c$. It follows that $\{x \in X \mid f(x) = g(x)\} = h^{-1}(\Delta(Y))$ is closed.

(b) Since $\{x \in X \mid f(x) = g(x)\}$ is closed and contains a dense subset of X, it is equal to X. Therefore $f = g$.

17. Suppose that, for every $x, y \in X$ with $x \neq y$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Given $x, y \in X$ with $x \neq y$, choose such a function f and disjoint open sets $U, V \subseteq \mathbb{R}$ such that $f(x) \in U$ and $f(y) \in V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint members of $\mathcal{T}$ with $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. This shows that $\mathcal{T}$ is Hausdorff.

Now suppose that $\mathcal{T}$ is Hausdorff, and let $x, y \in X$ be distinct. There exists $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$. Clearly $\varnothing \neq U \neq X$, so there exist $U_1, U_2, \ldots, U_n \subseteq \mathbb{R}$ and $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that $x \in \cap_{k=1}^n f_k^{-1}(U_k) \subseteq U$. It follows that $y \notin f_k^{-1}(U_k)$ for some $k \in \{1, 2, \ldots, n\}$, and hence $f_k(x) \neq f_k(y)$.

22. If $x_0 \in X$, then $0 \leq \rho(f_n(x_0), f_m(x_0)) \leq \sup_{x \in X} \rho(f_n(x), f_m(x))$ for all $m, n \in \mathbb{N}$. Therefore $(f_n(x_0))_{n=1}^\infty$ is a Cauchy sequence in the complete metric space (Y, ρ), which has a limit $f(x_0)$. This defines a map $f : X \to Y$. Given $\varepsilon \in (0, \infty)$ there exists $N \in \mathbb{N}$ such that $\sup_{x \in X} \rho(f_n(x), f_m(x)) < \frac{\varepsilon}{2}$ for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$. If $x \in X$ and $n \in \mathbb{N}$ with $n \geq N$ then $\rho(f_m(x), f(x)) < \frac{\varepsilon}{2}$ for some $m \in \mathbb{N}$ with $m \geq n$, so $\rho(f_n(x), f(x)) < \varepsilon$. This implies that $\sup_{x \in X} \rho(f_n(x), f(x)) \leq \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq N$, so $(\sup_{x \in X} \rho(f_n(x), f(x)))_{n=1}^\infty$ converges to 0.

Suppose $g : X \to Y$ and $(\sup_{x\in X} \rho(f_n(x), g(x)))_{n=1}^{\infty}$ converges to 0. If $\varepsilon \in (0,\infty)$ and $x \in X$ then $\rho(f_N(x), f(x)) < \frac{\varepsilon}{2}$ and $\rho(f_N(x), g(x)) < \frac{\varepsilon}{2}$ for some $N \in \mathbb{N}$, which implies that $\rho(f(x), g(x)) < \varepsilon$. Therefore $\rho(f(x), g(x)) = 0$, so $f(x) = g(x)$ and hence $f = g$. This shows that f is unique.

Now suppose that f_n is continuous for all $n \in \mathbb{N}$. Let $x_0 \in X$ and suppose that $U \subseteq Y$ is a neighbourhood of $f(x_0)$. There exists $\varepsilon \in (0,\infty)$ such that $B_\varepsilon(f(x_0)) \subseteq U$, and there exists $N \in \mathbb{N}$ such that $\sup_{x\in X} \rho(f_N(x), f(x)) < \frac{\varepsilon}{3}$. Since f_N is continuous, $V := f_N^{-1}(B_{\varepsilon/3}(f_N(x_0)))$ is an open neighbourhood of x_0. If $x \in V$ then

$$\rho(f(x), f(x_0)) \le \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(x_0)) + \rho(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and hence $f(V) \subseteq B_\varepsilon(f(x_0)) \subseteq U$. This implies that $f^{-1}(U)$ is a neighbourhood of x_0, so f is continuous.

23. Let $A \subseteq \mathbb{R}$ be closed $a \in \mathbb{R}$, $b \in (a,\infty)$ and $f \in C(A, [a,b])$. We aim to extend f to some $F \in C(\mathbb{R}, [a,b])$. Since A^c is open, it is a disjoint union of open intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$. At most two of these are unbounded, and on these we let F be constant, equal to the value of f at the finite endpoint. On bounded intervals (a_n, b_n), we let F be the linear function from $(a_n, f(a_n))$ to $(b_n, f(b_n))$. By construction F has the same image as f, and is continuous on A^c.

    Given $x \in A$, we will show that F is right continuous at x; the proof of left continuity is similar. For each $\varepsilon \in (0,\infty)$, there exists $\delta \in (0,\infty)$ such that $|f(x) - f(y)| < \varepsilon$ for all $y \in (x, x+\delta) \cap A$. If $(x, x+\delta) \subseteq A$ then we are done. Otherwise $(x, x+\delta)$ meets (a_n, b_n) for some $n \in \mathbb{N}$; this implies that $a_n = x$ or $a_n \in (x, x+\delta)$. If $a_n = x$ we are done, since F is continuous on $[a_n, b_n]$. Otherwise $|F(x) - F(y)| < \varepsilon$ for all $y \in (x, a_n)$: if $y \in A$ we know this already, and if $y \in (a_m, b_m)$ for some $m \in \mathbb{N}$ then $F(y)$ is between $f(a_m)$ and $f(b_m)$, which are both within ε of $f(x) = F(x)$ because $x \le a_m < b_m < a_n < x + \delta$ and $a_m, b_m \in A$. Therefore F is right continuous at x, and we are done.

24. If X is normal, then by Urysohn's lemma and the Tietze extension theorem, it satisfies the conclusions of Urysohn's lemma and the Tietze extension theorem. Conversely, if X satisfies the conclusion of Urysohn's lemma, we claim that X is normal. To this end, let A and B be disjoint closed subsets of X. There exists $f \in C(X, [0,1])$ such that $f|_A = 0$ and $f|_B = 1$. Note that $f^{-1}((-\infty, \frac{1}{37}))$ and $f^{-1}((\frac{37}{42}, \infty))$ are disjoint open sets which contain A and B respectively. Therefore X is normal. Finally, suppose that X satisfies the conclusion of the Tietze extension theorem. We need to show that X is normal; by the above it suffices to show that X satisfies the conclusion of Urysohn's lemma. To this end, let A and B be disjoint closed subsets of X. Note that $A \cup B$ is closed, and that A and B are open in $A \cup B$. Therefore $\chi_B \in C(A \cup B, [0,1])$, and there exists $f \in C(X, [0,1])$ extending χ_B, as required.

26. (a) As is often the case, it is easier to talk about disconnected spaces. We need to show that, if $f(X)$ is disconnected, then X is disconnected. Since $f(X)$ is disconnected, it has a nonempty proper subset A which is both open and closed (clopen). Note that $A = U \cap f(X)$ for some open set $U \subseteq Y$, and $A = C \cap f(X)$ for some closed set $C \subseteq Y$, by definition of the relative topology on $f(X)$. Therefore $f^{-1}(A) = f^{-1}(U) = f^{-1}(C)$ is a nonempty proper clopen subset of X. This implies that X is disconnected.

    (b) Again we prove the contrapositive: if X is disconnected, then it is not arcwise connected (usually this is called path connected). Let A be a nonempty proper clopen subset of X. There exists $x_0 \in A$ and $x_1 \in A^c$, but there is no path joining these points: if there was such a path, say f, then $f([0,1]) \cap A$ is a nonempty proper clopen subset of $f([0,1])$, which is impossible by part (a).

    (c) By part (a) the spaces $X^+ := X \cap ((0,\infty) \times \mathbb{R})$ and $X^- := X \cap ((-\infty, 0) \times \mathbb{R})$ are connected. If A is a clopen subset of X, we need to show that $A = \varnothing$ or $A = X$. Without loss of generality $(0,0) \in A$ (otherwise, take A^c). Clearly, there is a sequence in X^+ which converges to $(0,0)$. Since A is a neighbourhood of $(0,0)$, it follows that

$A \cap X^+$ is a nonempty clopen subset of X^+. This implies that $A \cap X^+ = X^+$. Similarly, $A \cap X^- = X^-$. Therefore $A = X$, which shows that X is connected.

If X is path connected, there should be a path f from $(0,0)$ to $(\frac{1}{\pi}, 0)$. Define $t_0 := \sup f^{-1}(\{(0,0)\})$ and note that $f(t_0) = (0,0)$ because $f^{-1}(\{(0,0)\})$ is compact. Since f is continuous, there exists $t_1 \in (t_0, 1)$ such that $f(t) \in B_{\frac{1}{2}}(0,0)$ for all $t \in [t_0, t_1]$. Note that $f(t_1) \in Y$, so $\pi_1(f(t_1)) \neq 0$, where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the x-axis. In particular, there is a point $(x, 1)$ of Y such that x is between $\pi_1(f(t_1))$ and 0. By part (a) $\pi_1(f([t_0, t_1]))$ is connected, so it contains x (if not, then $(-\infty, x) \cap \pi_1(f([t_0, t_1]))$ is a nonempty proper clopen subset) and hence $f([t_0, t_1])$ contains $(x, 1)$. This contradicts the fact that $f([t_0, t_1]) \subseteq B_{\frac{1}{2}}(0,0)$.

27. If $X = \varnothing$ then X is connected. Otherwise choose $x \in X$ and let $C \subseteq X$ be the connected component of x. Suppose that $C \neq X$. Then there exists $y \in C^c$, and C is closed (by the previous homework), so there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ and $U_{\alpha_1} \subseteq X_{\alpha_1}, U_{\alpha_2} \subseteq X_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq X_{\alpha_n}$ open such that $y \in \cap_{k=1}^n \pi_{\alpha_k}^{-1}(U_{\alpha_k}) \subseteq C^c$. Define $z_k \in X$ by

$$\pi_\alpha(z_k) := \begin{cases} \pi_\alpha(y), & \alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \\ \pi_\alpha(x), & \alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \end{cases}$$

for each $k \in \{0, 1, \ldots, n\}$, so that $z_0 = x$ and $z_n \in \cap_{k=1}^n \pi_{\alpha_k}^{-1}(U_{\alpha_k})$. Given $k \in \{1, 2, \ldots, n\}$ define $i_k : X_{\alpha_k} \to X$ by

$$\pi_\alpha(i_k(p)) := \begin{cases} p, & \alpha = \alpha_k \\ \pi_\alpha(z_k), & \alpha \neq \alpha_k \end{cases},$$

so that $\pi_\alpha \circ i_k$ is continuous for all $\alpha \in A$ and i_k is continuous. In particular $i_k(X_{\alpha_k})$ is connected, and it contains both z_k and z_{k-1}. Since $z_0 = x$ this implies that $i_1(X_{\alpha_1}) \subseteq C$, and by induction it follows that $z_n \in i_n(X_{\alpha_n}) \subseteq C$. But $z_n \in C^c$, which is a contradiction. Therefore $C = X$, so X is connected.

32. Suppose X is Hausdorff and let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X which converges to $x \in X$. If $y \in X$ and $x \neq y$, there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. There exists $\beta \in A$ such that $x_\alpha \in U$ for all $\alpha \in A$ with $\alpha \gtrsim \beta$. If $\gamma \in A$ then there exists $\delta \in A$ such that $\delta \gtrsim \beta$ and $\delta \gtrsim \gamma$, which implies that $\langle x_\alpha \rangle_{\alpha \in A}$ is not eventually in V because $x_\delta \notin V$. Therefore x is the only limit point of $\langle x_\alpha \rangle_{\alpha \in A}$.

Conversely, suppose that X is not Hausdorff, and choose $x, y \in X$ distinct with no disjoint open neighbourhoods. For each pair $(N_x, N_y) \in \mathcal{N}_x \times \mathcal{N}_y$ we may choose $x_{N_x, N_y} \in N_x \cap N_y$, because if N_x and N_y were disjoint, they would contain disjoint open neighbourhoods of x and y respectively. If $N_x \in \mathcal{N}_x$, choose $N_y \in \mathcal{N}_y$ and note that $x_{O_x, O_y} \in O_x \cap O_y \subseteq N_x$ for all $(O_x, O_y) \in \mathcal{N}_x \times \mathcal{N}_y$ such that $(O_x, O_y) \gtrsim (N_x, N_y)$. This implies that $\langle x_{N_x, N_y} \rangle_{(N_x, N_y) \in \mathcal{N}_x \times \mathcal{N}_y}$ converges to x, and a similar argument shows that it also converges to y.

34. If $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in X which converges to $x \in X$, and $f \in \mathcal{F}$, then f is continuous so $\langle f(x_\alpha) \rangle_{\alpha \in A}$ converges to $f(x)$.

Conversely, let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X and suppose there exists $x \in X$ such that $\langle f(x_\alpha) \rangle_{\alpha \in A}$ converges to $f(x)$ for all $f \in \mathcal{F}$. Given a neighbourhood $U \subseteq X$ of x, there exist $f_1, f_2, \ldots, f_n \in \mathcal{F}$ and $U_1 \subseteq f_1(X), U_2 \subseteq f_2(X), \ldots, U_n \subseteq f_n(X)$ open such that $x \in \cap_{k=1}^n f_k^{-1}(U_k) \subseteq U$. For each $k \in \{1, 2, \ldots, n\}$ there exists $\beta_k \in A$ such that $f_k(x_\alpha) \in U_k$ for all $\alpha \in A$ with $\alpha \gtrsim \beta_k$. By induction there exists an upper bound β for $\{\beta_1, \beta_2, \ldots, \beta_n\}$, and it follows that $x_\alpha \in \cap_{k=1}^n f_k^{-1}(U_k) \subseteq U$ for all $\alpha \in A$ with $\alpha \gtrsim \beta$. Therefore $\langle x_\alpha \rangle_{\alpha \in A}$ converges to x.

36. Suppose there exists a topology $\mathcal{T}$ on X in which convergence corresponds to pointwise almost everywhere convergence. For each $n \in \mathbb{N} \cup \{0\}$ and $k \in \{0, 1, \ldots, 2^n - 1\}$ define $f_{2^n + k} := \chi_{[2^{-n}k, 2^{-n}(k+1)]}$. Since it takes on each of the values 0

and 1 infinitely many times, $(f_n(x))_{n=1}^\infty$ fails to converge for all $x \in [0,1]$. In particular, $(f_n)_{n=1}^\infty$ does not converge to 0 in $(X, \mathcal{T})$. Hence there exists $U \in \mathcal{T}$ such that $0 \in U$ with the property that for each $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $n \geq N$ and $f_n \notin U$. Define $n_0 := 0$, and for each $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ so that $n_k > n_{k-1}$ and $f_{n_k} \notin U$. Clearly $(f_{n_k})_{k=1}^\infty$ converges to 0 in L^1 (as does $(f_n)_{n=1}^\infty$), so it has a subsequence $(f_{n_{k_j}})_{j=1}^\infty$ which converges to 0 pointwise almost everywhere. This is impossible because $f_{n_{k_j}} \notin U$ for all $j \in \mathbb{N}$.

38. Suppose that $\mathcal{T} \subseteq \mathcal{T}'$. If $x, y \in X$ and $x \neq y$, there exist disjoint sets $U, V \in \mathcal{T} \subseteq \mathcal{T}'$ such that $x \in U$ and $y \in V$. Therefore $(X, \mathcal{T}')$ is Hausdorff. Suppose that $(X, \mathcal{T}')$ is compact, and let $V \in \mathcal{T}'$. Then V^c is closed in $\mathcal{T}'$, hence compact relative to $\mathcal{T}'$. The identity map from $(X, \mathcal{T}')$ to $(X, \mathcal{T})$ is continuous, so V^c is compact relative to $\mathcal{T}$. Since $\mathcal{T}$ is Hausdorff V^c is closed in $\mathcal{T}$, so $V \in \mathcal{T}$ and hence $\mathcal{T} = \mathcal{T}'$. In particular, if $\mathcal{T} \subset \mathcal{T}'$ then $(X, \mathcal{T}')$ is not compact.

    Conversely, suppose that $\mathcal{T}' \subset \mathcal{T}$. The identity map from $(X, \mathcal{T})$ to $(X, \mathcal{T}')$ is continuous, so $(X, \mathcal{T}')$ is compact. Therefore $(X, \mathcal{T}')$ is not Hausdorff, by the previous argument.

39. Suppose a space X has a countable open cover $\{U_n\}_{n=1}^\infty$ with no finite subcover. For each $n \in \mathbb{N}$ there exists $x_n \in (\cup_{k=1}^n U_k)^c$. Let $\langle x_{n_k}\rangle_{k=1}^\infty$ be a subsequence of $\langle x_k\rangle_{k=1}^\infty$ and let $x_0 \in X$. Then $x_0 \in U_m$ for some $m \in \mathbb{N}$, and $x_{n_k} \notin U_m$ for sufficiently large $k \in \mathbb{N}$. This implies that $\langle x_{n_k}\rangle_{k=1}^\infty$ does not converge, so X is not sequentially compact.

40. Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in X, and for each $n \in \mathbb{N}$ define $E_n := \{x_k\}_{k=n}^\infty$. If $\cap_{n=1}^\infty \overline{E}_n = \varnothing$, then $\{(\overline{E}_n)^c\}_{n=1}^\infty$ is a countable open cover of X, which has a finite subcover $\{(\overline{E}_{n_1})^c, (\overline{E}_{n_2})^c, \ldots, (\overline{E}_{n_m})^c\}$. It follows that

    $$\cap_{k=1}^m E_{n_k} \subseteq \cap_{k=1}^m \overline{E}_{n_k} = (\cup_{k=1}^m (\overline{E}_{n_k})^c)^c = X^c = \varnothing,$$

    which is a contradiction because $x_N \in \cap_{k=1}^m E_{n_k}$ for $N := \max\{n_k\}_{k=1}^m$. Hence there exists $x_0 \in \cap_{n=1}^\infty \overline{E}_n$. If $U \subseteq X$ is a neighbourhood of x_0 and $n \in \mathbb{N}$ then $U \setminus \{x_0\}$ meets E_n, so there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x_k \in U$. This implies that x_0 is a cluster point of $\langle x_n\rangle_{n=1}^\infty$. By exercise 7 (see above), it follows that $\langle x_n\rangle_{n=1}^\infty$ has a convergent subsequence provided that X is first countable.

43. Let $\langle a_{n_k}\rangle_{k=1}^\infty$ be a subsequence of $\langle a_k\rangle_{k=1}^\infty$, and define $x := \sum_{k=1}^\infty (1+(-1)^k)2^{-(n_k+1)}$. Then $x \in [0,1)$ (in fact $x \in [0, \frac{1}{4}]$) and $a_{n_k}(x) = \frac{1}{2}(1+(-1)^k)$ for all $k \in \mathbb{N}$. The sequence $\langle a_{n_k}(x)\rangle_{k=1}^\infty = \langle 0,1,0,1,\ldots\rangle$ does not converge, so $\langle a_{n_k}\rangle_{k=1}^\infty$ does not converge in $[0,1]^{[0,1)}$ and hence $\langle a_k\rangle_{k=1}^\infty$ has no convergent subsequence.

44. If $\{U_n\}_{n=1}^\infty$ is a (countable) open cover of $f(X)$, then $\{f^{-1}(U_n)\}_{n=1}^\infty$ is an open cover of X, which has a finite subcover, say $\{f^{-1}(U_n)\}_{n=1}^N$. It follows that $\{U_n\}_{n=1}^N$ covers $f(X)$, which shows that $f(X)$ is countably compact.

45. Suppose X is countably compact. If $f \in C(X)$, then $f(X)$ is countably compact by Exercise 44. Since $\mathbb{C}$ is first countable, it follows that $f(X)$ is sequentially compact by Exercise 40, and hence bounded by Heine-Borel.

    Conversely, suppose that X has a countable open cover $\{U_n\}_{n=1}^\infty$ with no finite subcover. For each $n \in \mathbb{N}$ there exists $x_n \in (\cup_{k=1}^n U_k)^c$. Define $C := \{x_n\}_{n=1}^\infty$, and let $K \subseteq C$. If $x_0 \in X$ then $x_0 \in U_n$ for some $n \in \mathbb{N}$. Since X is T_1, for each $k \in \{1, 2, \ldots, n\}$ there is an open neighbourhood V_k of x_0 such that $x_k \notin V_k \setminus \{x_0\}$ (if $x_k = x_0$ just set $V_k := X$). It follows that $U_n \cap (\cap_{k=1}^n V_k)$ is a neighbourhood of x_0 which is disjoint from $K \setminus \{x_0\}$, so $x_0 \notin \mathrm{acc}(K)$. Therefore $\mathrm{acc}(K) = \varnothing$, so every subset of C is closed. If $n \in \mathbb{N}$ then $x_n \in U_m$ for some $m \in \mathbb{N}$, and $\langle x_k\rangle_{k=1}^\infty$ eventually lies outside U_m. Hence we may define $f : C \to \mathbb{N}$ by $f(c) := \max\{k \in \mathbb{N} \mid x_k = c\}$. Note that f is continuous and unbounded, because every subset of C is closed and $f(x_n) \geq n$ for all $n \in \mathbb{N}$. By Tietze's extension theorem, there exists an extension $F \in C(X)$ of f, which is unbounded and hence $C(X) \neq BC(X)$.

49. (a) If $x \in E$, then x has a compact neighbourhood $N \subseteq E$. This contains an open neighbourhood $U \subseteq N$ of x, and $U = U \cap E$ is relatively open, so N is a neighbourhood of x in the relative topology. It is actually a compact neighbourhood, because the topologies on N relative to E and X are the same. Therefore E is locally compact.

(b) Let $x \in E$ and choose a compact neighbourhood $N \subseteq E$ of x in the relative topology. As N is a compact subspace of the Hausdorff space X, it is closed. Let $U \subseteq N$ be a relatively open neighbourhood of x, so that $U = V \cap E$ for some open $V \subseteq X$. By Exercise 13, $x \in V \subseteq \overline{V} = \overline{V \cap E} \subseteq N \subseteq E$, which shows that E is open.

(c) Suppose E is locally compact in the relative topology. Since $\overline{E} \subseteq X$ is closed, it is compact and $\overline{E}$ is a compact Hausdorff space. The relative topologies on E inherited from $\overline{E}$ and X are the same, so E is locally compact in the relative topology inherited from $\overline{E}$. If $C \subseteq \overline{E}$ is relatively closed and $E \subseteq C$, then $C = D \cap \overline{E}$ for some closed $D \subseteq X$, so C is closed and hence $C = \overline{E}$. This implies that E is dense in $\overline{E}$, so E is relatively open in $\overline{E}$ by part (b) with X replaced by $\overline{E}$.

Conversely, suppose that E is relatively open in $\overline{E}$. Again, note that $\overline{E}$ is a compact Hausdorff space, in which case E is locally compact in the relative topology inherited from $\overline{E}$, by part (a). This is the same as the relative topology inherited from X, so E is also locally compact in that topology.

51. Suppose ϕ is proper, and extend it to a map $X^* \to Y^*$. Let $U \subseteq Y^*$ be open. If $\infty_Y \notin U$ then $\phi^{-1}(U)$ is open, because $\phi|_X$ is continuous. Otherwise $U^c \subseteq Y$ is compact, so $\phi^{-1}(U^c)$ is compact and hence $\phi^{-1}(U) = \phi^{-1}(U^c)^c$ is open (as X^* is Hausdorff, or because $\infty_X \in \phi^{-1}(U)$). Therefore ϕ is continuous.

Conversely, suppose that ϕ extends continuously to a map $X^* \to Y^*$. If $K \subseteq Y$ is compact, then K^c is open in Y^*, so $\phi^{-1}(K^c)$ is open in X^*. Since $\infty_X \in \phi^{-1}(K^c)$, it follows that $\phi^{-1}(K) = \phi^{-1}(K^c)^c$ is compact. Thus ϕ is proper.

54. (a) If $K \subseteq \mathbb{Q}$ is a neighbourhood of 0, then $(-\varepsilon, \varepsilon) \cap \mathbb{Q} \subseteq K$ for some $\varepsilon \in (0, \infty) \cap \mathbb{Q}$. Clearly $\frac{\varepsilon}{\sqrt{2}} \in \overline{K} \setminus K$, so K is not a closed subset of $\mathbb{R}$; in particular it is not compact. Therefore 0 has no compact neighbourhood in $\mathbb{Q}$.

(b) Define $K := \{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty}$, and note that $K^c = \cup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n}) \cup (-\infty, 0) \cup (1, \infty)$ is open, so K is closed (in $\mathbb{R}$) and bounded, hence compact. For each $n \in \mathbb{N}$ let $f_n : \mathbb{Q} \to \mathbb{C}$ be the indicator function of $\{\frac{1}{n}\}$. The sequence $\langle f_n \rangle_{n=1}^{\infty}$ converges to 0 pointwise, but not uniformly on K.

56. (a) If $t \in (0, \infty)$ then $\Phi'(t) = (t+1)^{-2} > 0$, so Φ is strictly increasing on $[0, \infty)$ by the mean value theorem. Since $\Phi(t) < 1$ for all $t \in [0, \infty)$, it follows that Φ is strictly increasing. If $s, t \in [0, \infty)$ then

$$\Phi(t+s) = \frac{t+s}{1+t+s} = \frac{t}{1+t+s} + \frac{s}{1+t+s} \leq \frac{t}{1+t} + \frac{s}{1+s} = \Phi(s) + \Phi(t),$$

and the same clearly holds if $s = \infty$ or $t = \infty$.

(b) If $y \in Y$ then $\Phi(\rho(y, y)) = \Phi(0) = 0$. Conversely, if $x, y \in Y$ and $\Phi(\rho(x, y)) = 0$ then $\rho(x, y) = 0$ (as Φ is injective) and hence $x = y$. Clearly $\Phi \circ \rho$ is nonnegative and bounded. If $x, y, z \in Y$ then $\Phi(\rho(x, y)) = \Phi(\rho(y, x))$ and $\Phi(\rho(x, z)) \leq \Phi(\rho(x, y) + \rho(y, z)) \leq \Phi(\rho(x, y)) + \Phi(\rho(y, z))$. Therefore $\Phi \circ \rho$ is a metric.

If $y \in Y$ and $r \in (0, \infty)$ then

$$B_r^{\rho}(y) = \{x \in Y \mid \rho(x, y) < r\} = \{x \in Y \mid \Phi(\rho(x, y)) < \Phi(r)\} = B_{\Phi(r)}^{\Phi\circ\rho}(y).$$

Moreover, if $r < 1$ then $r = \Phi(\frac{r}{1-r})$ and hence

$$B_r^{\Phi\circ\rho}(y) = \{x \in Y \mid \Phi(\rho(x, y)) < \Phi(\tfrac{r}{1-r})\} = \{x \in Y \mid \rho(x, y) < \tfrac{r}{1-r}\} = B_{\frac{r}{1-r}}^{\rho}(y).$$

Otherwise $B_r^{\Phi\circ\rho}(y) = Y$. This shows that ρ and $\Phi \circ \rho$ define the same topologies on Y.

(c) The proof is the same as part (b), except that

$$B_1(g) = \{f \in \mathbb{C}^X \mid \rho(f, g) < 1\} = \{f \in \mathbb{C}^X \mid \sup_{x \in X} |f(x) - g(x)| < \infty\}$$

for all $g \in \mathbb{C}^X$. These sets are still open in the topology of uniform convergence, as required.

(d) It is routine to check that ρ is a metric. Let $r \in (0, \infty)$ and $f \in \mathbb{C}^X$. For each $g \in B_r(f)$ there exists $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(g) \subseteq B_r(f)$. Choose $m, N \in \mathbb{N}$ so that $\Phi(m^{-1}) < \frac{\varepsilon}{2}$ and $\sum_{n=N}^\infty 2^{-n} < \frac{\varepsilon}{2}$. Note that

$$\{h \in \mathbb{C}^X \mid \sup_{x \in \overline{U}_N} |h(x) - g(x)| < m^{-1}\} \subseteq \left\{h \in \mathbb{C}^X \;\middle|\; \sum_{n=1}^{N-1} 2^{-n}\Phi\left(\sup_{x \in \overline{U}_n} |h(x) - g(x)|\right) < \frac{\varepsilon}{2}\right\} \subseteq B_\varepsilon(g).$$

This shows that $B_r(f)$ is open in the topology of uniform convergence on compact sets. Conversely, let $f \in \mathbb{C}^X$ and $m, n \in \mathbb{N}$. For each $g \in \mathbb{C}^X$ with $\sup_{x \in \overline{U}_n} |g(x) - f(x)| < m^{-1}$, there exists $\varepsilon \in (0, \infty)$ such that

$$\{h \in \mathbb{C}^X \mid \sup_{x \in \overline{U}_n} |h(x) - g(x)| < \varepsilon\} \subseteq \{h \in \mathbb{C}^X \mid \sup_{x \in \overline{U}_n} |h(x) - f(x)| < m^{-1}\}.$$

It is clear that $B_{2^{-n}\Phi(\varepsilon)}(g)$ is contained in the former set. This shows that the latter set is open in the metric topology, so the two topologies are equivalent.

57. (a) Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X, and choose a sequence $\langle E_n \rangle_{n=1}^\infty$ of precompact open subsets of X such that $\overline{E}_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$ and $\cup_{n=1}^\infty E_n = X$. Set $E_0 := \varnothing$ and for each $n \in \mathbb{N}$ define $\mathcal{U}_n := \{U_\alpha \cap (E_{n+2} \setminus \overline{E}_{n-1})\}_{\alpha \in A}$, so that $\mathcal{U}_n$ is an open cover of the compact set $\overline{E}_{n+1} \setminus E_n$ (or $\overline{E}_2$ in the case $n = 1$), having a finite subcover $\mathcal{V}_n$. Write $\cup_{n=1}^\infty \mathcal{V}_n = \{V_n\}_{n=1}^\infty$, and note that this cover is a refinement of $\{U_\alpha\}_{\alpha \in A}$. We claim it is locally finite. Indeed, if $x \in X$ then $x \in E_{n+2} \setminus \overline{E}_{n-1}$ for some $n \in \mathbb{N}$, which is an open neighbourhood of x that meets no members of $\mathcal{V}_m$ for all $m \in \mathbb{N}$ with $|m - n| \geq 3$. If $n \in \mathbb{N}$ then $\overline{V}_n$ is a closed subset of $\overline{E}_{m+2}$ for some $m \in \mathbb{N}$, and is therefore compact.

For each $x \in X$ there exists $n \in \mathbb{N}$ such that $x \in V_n$, and x has a compact neighbourhood $N_x \subseteq V_n$. Given $n \in \mathbb{N}$, there is a finite set $Y_n \subseteq X$ such that $\overline{V}_n \subseteq \cup_{y \in Y_n} N_y^\circ$. From above it is clear that V_n meets only finitely many members of $\{V_m\}_{m=1}^\infty$, say $\{V_m\}_{m \in J_n}$ for some finite index set $J_n \subseteq \mathbb{N}$. Define $Z_n := \{y \in \cup_{m \in J_n} Y_m \mid N_y \subseteq V_n\}$ and $W_n := \cup_{y \in Z_n} N_y^\circ$. Since Z_n is finite $\overline{W}_n \subseteq \cup_{y \in Z_n} N_y \subseteq V_n$. Moreover, if $x \in X$ then there exists $n \in \mathbb{N}$ such that $x \in V_n \subseteq \cup_{y \in Y_n} N_y^\circ$, so $x \in N_y^\circ$ for some $y \in Y_n$. By definition $N_y \subseteq V_m$ for some $m \in \mathbb{N}$, and $x \in V_n \cap V_m$ so $n \in J_m$, implying that $y \in Z_m$ and hence $x \in W_m$. This shows that $\{W_n\}_{n=1}^\infty$ is an open cover of X. Since it is a refinement of $\{V_n\}_{n=1}^\infty$, it is a locally finite refinement of $\{U_\alpha\}_{\alpha \in A}$. Note also that W_n is precompact for each $n \in \mathbb{N}$ (as $\overline{W}_n \subseteq \overline{V}_n$).

(b) For each $n \in \mathbb{N}$, by Urysohn's lemma there exists $f_n \in C_c(X, [0, 1])$ such that $f_n(\overline{W}_n) = \{1\}$ and $f_n(V_n^c) = 0$, where $\{W_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ are the covers constructed in part (a). Since $\{V_n\}_{n=1}^\infty$ is locally finite, each $x \in X$ has an open neighbourhood on which $f := \sum_{n=1}^\infty f_n$ is well-defined and continuous. Note that $f \geq 1$ because $\{W_n\}_{n=1}^\infty$ covers X, and in particular $g_n := f_n/f$ is a well-defined member of $C_c(X, [0, 1])$ for each $n \in \mathbb{N}$. It is clear that $\{g_n\}_{n=1}^\infty$ is a partition of unity subordinate to $\mathcal{U}$.

58. Let $K \subseteq \prod_{\alpha \in A} X_\alpha$ be closed, and suppose there exists $x \in K^\circ$. Then there exists a finite subset $B \subseteq A$ and open sets $U_\beta \subseteq X_\beta$ for each $\beta \in B$ such that $x \in \cap_{\beta \in B} \pi_\beta^{-1}(U_\beta) \subseteq K$. Since B is finite, X_α is noncompact for some $\alpha \in A \setminus B$. Hence there exists an open cover $\{V_\gamma\}_{\gamma \in \Gamma}$ of X_α which has no finite subcover. If $\{\pi_\alpha^{-1}(V_\delta)\}_{\delta \in \Delta}$ covers K, for some $\Delta \subseteq \Gamma$, then $\{V_\delta\}_{\delta \in \Delta}$ covers X_α. Indeed, if $y_\alpha \in X_\alpha$ then the point $y \in \prod_{\alpha \in A} X_\alpha$ defined by $\pi_\alpha(y) = y_\alpha$

and $\pi_\beta(y) = \pi_\beta(x)$ for all $\beta \in A \setminus \{\alpha\}$ lies in K, so $y \in \pi_\alpha^{-1}(V_\delta)$ and hence $y_\alpha \in V_\delta$ for some $\delta \in \Delta$. This implies that $\{\pi_\alpha^{-1}(V_\gamma)\}_{\gamma\in\Gamma}$ is an open cover of K with no finite subcover. Therefore K is not compact, which shows that every closed compact subset of $\prod_{\alpha\in A} X_\alpha$ has empty interior.

59. Let X and Y be locally compact spaces. If $(x, y) \in X \times Y$, then x and y have compact neighbourhoods $K \subseteq X$ and $L \subseteq Y$. By Tychonoff's theorem $K \times L$ is a compact subset of $X \times Y$, and it is a neighbourhood of (x, y) (by the definition of the product topology). Therefore $X \times Y$ is locally compact; by induction a finite product of locally compact spaces is locally compact.

60. Let $\{X_n\}_{n=1}^\infty$ be a collection of sequentially compact spaces, and let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\prod_{n=1}^\infty X_n$. Starting with $\langle x_n^0\rangle_{n=1}^\infty := \langle x_n\rangle_{n=1}^\infty$, define for each $m \in \mathbb{N}$ a subsequence $\langle x_n^m\rangle_{n=1}^\infty$ of $\langle x_n^{m-1}\rangle_{n=1}^\infty$ such that $\langle \pi_m(x_n^m)\rangle_{n=1}^\infty$ converges to $x^m \in X_m$. Define $x \in \prod_{n=1}^\infty X_n$ by $\pi_n(x) = x^n$ for all $n \in \mathbb{N}$, and let $U \subseteq \prod_{n=1}^\infty X_n$ be a neighbourhood of x. There exists $N \in \mathbb{N}$ and open subsets $U_n \subseteq X_n$ for each $n \in \{1, 2, \ldots, N\}$ such that $x \in \cap_{n=1}^N \pi_n^{-1}(U_n) \subseteq U$. Given $m \in \{1, 2, \ldots, N\}$, $\langle x_n^n\rangle_{n=m}^\infty$ is a subsequence of $\langle x_n^m\rangle_{n=1}^\infty$, so there exists $N_m \in \mathbb{N}$ such that $\pi_m(x_n^n) = x^m$ for all $n \in \mathbb{N}$ with $n \geq N_m$. If $n \in \mathbb{N}$ and $n \geq \max\{N_m\}_{m=1}^N$ then $\pi_m(x_n^n) = x^m$ for all $m \in \{1, 2, \ldots, N\}$, and hence $x_n^n \in U$. This shows that $\langle x_n^n\rangle_{n=1}^\infty$ (a subsequence of $\langle x_n\rangle_{n=1}^\infty$) converges to x. Therefore $\prod_{n=1}^\infty X_n$ is sequentially compact.

63. If $f = 0$ then $Tf = 0 \in C([0, 1])$. Otherwise $\|f\|_u > 0$. Let $\varepsilon \in (0, \infty)$ and note that K is uniformly continuous (because $[0, 1]^2$ is compact). Hence there exists $\delta \in (0, \infty)$ such that $|K(z_1) - K(z_2)| < \varepsilon/\|f\|_u$ for all $z_1, z_2 \in [0, 1]^2$ with $|z_1 - z_2| < \delta$. It follows that

$$\begin{aligned} |Tf(x_1) - Tf(x_2)| &= \left|\int_0^1 K(x_1, y)f(y)\,dy - \int_0^1 K(x_2, y)f(y)\,dy\right| \\ &= \left|\int_0^1 (K(x_1, y) - K(x_2, y))f(y)\,dy\right| \\ &\leq \int_0^1 |K(x_1, y) - K(x_2, y)||f(y)|\,dy \\ &\leq \int_0^1 \frac{\varepsilon}{\|f\|_u}|f(y)|\,dy \\ &\leq \int_0^1 \frac{\varepsilon}{\|f\|_u}\|f\|_u\,dy \\ &= \int_0^1 \varepsilon\,dy \\ &= \varepsilon \end{aligned}$$

for all $x_1, x_2 \in [0, 1]$ with $|x_1 - x_2| < \delta$. This shows that $Tf \in C([0, 1])$. Now let $\varepsilon \in (0, \infty)$ and choose $\delta \in (0, \infty)$ such that $|K(z_1) - K(z_2)| < \varepsilon$ for all $z_1, z_2 \in [0, 1]^2$ with $|z_1 - z_2| < \delta$. Then $|K(z_1) - K(z_2)| < \varepsilon/\|f\|_u$ for all $f \in C([0, 1])$ with $0 < \|f\|_u \leq 1$ and $z_1, z_2 \in [0, 1]^2$ with $|z_1 - z_2| < \delta$, so $\{Tf \mid \|f\|_u \leq 1\}$ is equicontinuous by the above calculation and the fact that $|0 - 0| < \varepsilon$. Moreover, if $x \in [0, 1]$ and $f \in C([0, 1])$ with $\|f\|_u \leq 1$ then

$$|Tf(x)| = \left|\int_0^1 K(x, y)f(y)\,dy\right| \leq \int_0^1 |K(x, y)||f(y)|\,dy \leq \int_0^1 |K(x, y)|\,dy,$$

so $\{Tf \mid \|f\|_u \leq 1\}$ is pointwise bounded, and thus precompact by the Arzelà-Ascoli theorem.

64. Let $\varepsilon \in (0, \infty)$ and define $\delta := \sqrt[\alpha]{\varepsilon}$. If $f \in \mathcal{F} := \{f \in C(X) \mid \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ then

$$|f(x) - f(y)| \leq N_\alpha(f)\rho(x, y)^\alpha \leq \rho(x, y)^\alpha < \delta^\alpha = \varepsilon$$

for all $x, y \in X$ with $\rho(x, y) < \delta$. This implies that $\mathcal{F}$ is equicontinuous. Moreover, if $f \in \mathcal{F}$ and $x \in X$ then $|f(x)| \leq \|f\|_u \leq 1$, so $\mathcal{F}$ is pointwise bounded and hence precompact, by the Arzelà-Ascoli theorem. Now let $\langle f_n \rangle_{n=1}^\infty$ be a sequence in $\mathcal{F}$ which converges in $C(X)$ to some $f \in C(X)$. If $x \in X$ then $\langle f_n(x) \rangle_{n=1}^\infty$ is a sequence in $[-1, 1]$ which converges to $f(x)$, so $|f(x)| \leq 1$ and hence $\|f\|_u \leq 1$. Similarly, if $x, y \in X$ then $\langle f_n(x) - f_n(y) \rangle_{n=1}^\infty$ converges to $f(x) - f(y)$, and hence $|f(x) - f(y)| \leq \rho(x, y)^\alpha$. This implies that $f \in \mathcal{F}$, so $\mathcal{F}$ is closed, thus compact.

68. Define $\mathcal{E} := \{(x, y) \mapsto f(x)g(y) \mid f \in C(X), g \in C(Y)\}$. The collection $\mathcal{A}$ of finite sums of elements of $\mathcal{E}$ is the algebra generated by $\mathcal{E}$, because this algebra contains $\mathcal{A}$, and $\mathcal{A}$ is an algebra. Indeed, if $f_1, f_2, \ldots f_n, h_1, h_2, \ldots, h_m \in C(X)$ and $g_1, g_2, \ldots, g_n, k_1, k_2, \ldots, k_m \in C(Y)$, then
$$\left(\sum_{i=1}^n f_i g_i\right)\left(\sum_{j=1}^m h_j k_j\right) = \sum_{i=1}^n \sum_{j=1}^m f_i g_i h_j k_j = \sum_{i=1}^n \sum_{j=1}^m (f_i h_j)(g_i k_j) \in \mathcal{A}.$$
Note that $\mathcal{A}$ is closed under complex conjugation, because if $f_1, f_2, \ldots f_n \in C(X)$ and $g_1, g_2, \ldots, g_n \in C(Y)$ then
$$\overline{\sum_{i=1}^n f_i g_i} = \sum_{i=1}^n \overline{f_i g_i} = \sum_{i=1}^n \overline{f_i}\,\overline{g_i} \in \mathcal{A}.$$
Because complex conjugation is continuous, $\overline{\mathcal{A}}$ is also closed under complex conjugation. If $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $(x_1, y_1) \neq (x_2, y_2)$, then $x_1 \neq x_2$ or $y_1 \neq y_2$. In the former case, there exists $f \in C(X)$ with $f(x_1) \neq f(x_2)$ because X is normal, and hence $f \cdot 1 \in \mathcal{A}$ separates (x_1, y_1) from (x_2, y_2). The latter case is similar, and we conclude that $\overline{\mathcal{A}}$ separates points. Since $X \times Y$ is compact and Hausdorff, and $\mathcal{A}$ contains the constant functions, the complex Stone-Weierstraß theorem implies that $\overline{\mathcal{A}} = C(X \times Y)$.

69. Let $\mathcal{A}$ and $\mathcal{B}$ be the subalgebras of $C(X)$ and $C(X, \mathbb{R})$, respectively, generated by the coordinate maps and the constant function 1. Then $\mathcal{A} = \operatorname{span}_{\mathbb{C}}(\mathcal{B})$, because $\mathcal{A}$ contains $\mathcal{B}$ (note that $\mathcal{A} \cap C(X, \mathbb{R})$ is a subalgebra of $C(X, \mathbb{R})$) and $\operatorname{span}_{\mathbb{C}}(\mathcal{B})$ is an algebra. Indeed, if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in \mathbb{C}$ and $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m \in \mathcal{B}$ then
$$\left(\sum_{i=1}^n a_i f_i\right)\left(\sum_{j=1}^m b_j g_j\right) = \sum_{i=1}^n \sum_{j=1}^m a_i f_i b_j g_j = \sum_{i=1}^n \sum_{j=1}^m (a_i b_j)(f_i g_j) \in \operatorname{span}_{\mathbb{C}}(\mathcal{B}).$$
Therefore $\mathcal{A}$ is closed under complex conjugation, because if $a_1, a_2, \ldots, a_n \in \mathbb{C}$ and $f_1, f_2, \ldots f_n \in \mathcal{B}$ then
$$\overline{\sum_{i=1}^n a_i f_i} = \sum_{i=1}^n \overline{a_i f_i} = \sum_{i=1}^n \overline{a_i} f_i \in \mathcal{A}.$$
Because complex conjugation is continuous, $\overline{\mathcal{A}}$ is also closed under complex conjugation. If $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $\pi_\alpha(x_1) \neq \pi_\alpha(x_2)$ for some $\alpha \in A$, and hence $\overline{\mathcal{A}}$ separates points. By Tychonoff's theorem X is compact, and it is also Hausdorff because $[0, 1]$ is Hausdorff. Since $\mathcal{A}$ contains the constant functions, the complex Stone-Weierstraß theorem implies that $\overline{\mathcal{A}} = C(X)$.

70. (a) By definition $h(\mathcal{I}) = \bigcap_{f \in \mathcal{I}} f^{-1}(\{0\})$, which is closed because $\{0\}$ is closed in $\mathbb{R}$.

(b) Clearly $0 \in k(E)$. If $f, g \in k(E)$ and $a \in \mathbb{R}$ then $af(x) + g(x) = 0$ for all $x \in E$, so $af + g \in k(E)$. Moreover, if $f \in k(E)$ and $g \in C(X, \mathbb{R})$ then $f(x)g(x) = 0$ for all $x \in E$, so $fg \in k(E)$. This shows that $k(E)$ is an ideal of $C(X, \mathbb{R})$. For each $x \in E$ the coordinate map $\pi_x : C(X, \mathbb{R}) \to \mathbb{R}$ is (Lipschitz) continuous (relative to the uniform norm on $C(X, \mathbb{R})$), so $k(E) = \bigcap_{x \in E} e_x^{-1}(\{0\})$ is closed.

(c) Clearly $E \subseteq h(k(E))$, and hence $\overline{E} \subseteq h(k(E))$. If $x \in \overline{E}^c$ then, since X is normal, there exists $f \in C(X, \mathbb{R})$ such that $f(\overline{E}) = 0$ and $f(x) = 1$. It follows that $x \notin h(k(E))$, because $f \in k(E)$ but $f(x) \neq 0$. Thus $h(k(E)) = \overline{E}$.

(d) Clearly $\mathcal{I} \subseteq k(h(\mathcal{I}))$, and hence $\overline{\mathcal{I}} \subseteq k(h(\mathcal{I}))$. Define $U := X \setminus h(\mathcal{I})$, and let $x \in U$. Then $\{x\}$ and U^c are closed and disjoint, so there exist disjoint open neighbourhoods $V, W \subseteq X$ of x and U^c. It follows that $W^c \subseteq U$ is a compact neighbourhood of x, since X is compact and $V \subseteq W^c$. This shows that U is locally compact. Define $\mathcal{J} := \{ f|_U \mid f \in \overline{\mathcal{I}}\}$. If $f \in \overline{\mathcal{I}}$ and $\varepsilon \in (0, \infty)$ then $\{x \in U \mid |f|_U(x)| \geq \varepsilon\} = \{x \in X \mid f(x) \geq \varepsilon\}$ because $f(x) = 0$ for all $x \in h(\mathcal{I})$. Since this set is closed (thus compact), $f|_U \in C_0(U)$. Therefore $\mathcal{J}$ is a closed subalgebra of $C_0(U, \mathbb{R})$ (because function restriction respects pointwise addition and multiplication, and uniform limits). If $x, y \in U$ and $x \neq y$, there exists $f \in \mathcal{I}$ such that $f(x) \neq 0$. Since X is normal there exists $g \in C(X, \mathbb{R})$ such that $g(x) = 1$ and $g(y) = 0$. Then $gf \in \mathcal{I}$ and hence $(gf)|_U \in \mathcal{J}$, so $\mathcal{J}$ separates points. Moreover, there is no $x \in U$ such that $f(x) = 0$ for all $f \in \mathcal{J}$. By the Stone-Weierstraß theorem, it follows that $\mathcal{J} = C(U, \mathbb{R})$ or $\mathcal{J} = C_0(U, \mathbb{R})$, depending on whether U is closed or not. If $f \in k(h(\mathcal{I}))$ then $f|_U \in C_0(U, \mathbb{R})$ for the same reason that $\mathcal{J} \subseteq C_0(U, \mathbb{R})$, so $f|_U = g|_U$ for some $g \in \overline{\mathcal{I}}$, in which case $f = g$ because $f(x) = g(x) = 0$ for all $x \in h(\mathcal{I})$. This shows that $k(h(\mathcal{I})) \subseteq \overline{\mathcal{I}}$, and hence $k(h(\mathcal{I})) = \overline{\mathcal{I}}$.

(e) The previous two exercises show that k is a bijection from the closed subsets of X onto the closed ideals of $C(X, \mathbb{R})$.

76. Let $\mathcal{B}$ be a countable base for the topology on X. Since X is normal, for each $U, V \in \mathcal{B}$ such that $\overline{V} \subseteq U$, there exists $f_{U,V} \in C(X, [0, 1])$ which is 0 on U^c and 1 on $\overline{V}$. Clearly $\mathcal{F} := \{f_{U,V} \mid U, V \in \mathcal{B} \text{ and } \overline{V} \subseteq U\}$ is countable. Let $C \subset X$ be closed and let $x \in C^c$. There exists $U \in \mathcal{B}$ such that $x \in U \subseteq C^c$, and by normality there exist disjoint open sets $U', V' \subseteq X$ such that $U^c \subseteq U'$ and $x \in V'$. Then $x \in V \subseteq V'$ for some $V \in \mathcal{B}$, and $\overline{V} \subseteq \overline{V'} \subseteq (U')^c \subseteq U$. Now $C \subseteq U^c$, so $f_{U,V}(C) = \{0\}$, while $f_{U,V}(x) = 1$ because $x \in V$. This shows that $\mathcal{F}$ separates points and closed sets.

77. Clearly ρ maps into $[0, 1]$. If $x \in X$, then $\rho(x, x) = 0$ by definition. Given $x, y \in X$ with $x \neq y$, there exists $n \in \mathbb{N}$ such that $x_n \neq y_n$, in which case $\rho(x, y) \geq 2^{-n}\rho_n(x_n, y_n) > 0$. Clearly $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$. Moreover,

$$\rho(x,z) = \sum_{n=1}^{\infty} 2^{-n}\rho_n(x_n,z_n) \leq \sum_{n=1}^{\infty} 2^{-n}(\rho_n(x_n,y_n)+\rho(y_n,z_n)) = \sum_{n=1}^{\infty} 2^{-n}\rho_n(x_n,y_n)+\sum_{n=1}^{\infty} 2^{-n}\rho(y_n,z_n) = \rho(x,y)+\rho(y,z)$$

for all $x, y, z \in X$. This shows that ρ is a metric.

Let $n \in \mathbb{N}$ and $U \subseteq X_n$ be open. Given $x \in \pi_n^{-1}(U)$, there exists $r \in (0, \infty)$ such that $B_r(x_n) \subseteq U$. If $y \in B_{2^{-n}r}(x)$ then $\rho(x, y) < 2^{-n}r$, and in particular $\rho_n(x_n, y_n) < r$, which implies that $y_n \in U$ and hence $y \in \pi_n^{-1}(U)$. This shows that $\pi_n^{-1}(U)$ is open in (X, ρ), and hence every member of the product topology on X is open in (X, ρ).

Conversely, let $U \subseteq X$ be open in (X, ρ). Given $x \in U$, there exists $r \in (0, \infty)$ such that $B_r(x) \subseteq U$. Choose $N \in \mathbb{N}$ so that $2^{1-N} \leq r$. For each $n \in \{1, 2, \ldots, N\}$ define $U_n := B_{r/2}(x_n)$, and set $V := \cap_{n=1}^N \pi_n^{-1}(U_n)$. If $y \in V$ then

$$\rho(x,y) = \sum_{n=1}^{N} 2^{-n}\rho(x_n,y_n) + \sum_{n=N+1}^{\infty} 2^{-n}\rho(x_n,y_n) < \sum_{n=1}^{N} 2^{-n-1}r + \sum_{n=N+1}^{\infty} 2^{-n} < 2^{-1}r + 2^{-N} \leq r,$$

so $y \in U$. Since $x \in V$, this shows that U is open in the product topology. Therefore (X, ρ) has the product topology.