4. Note that $\|Tx - T_n x_n\| \leq \|Tx - Tx_n\| + \|Tx_n - T_n x_n\| \leq \|T\|\|x - x_n\| + \|T - T_n\|\|x_n\|$, and the limit as $n \to \infty$ of the right hand side is 0, so $\lim_{n\to\infty} T_n x_n = Tx$.

6. (a) Clearly $\|x\|_1 \geq 0$ for all $x \in \mathcal{X}$. If $\sum_{k=1}^n a_k e_k \in \mathcal{X}$ is non-zero then $a_m \neq 0$ for some $m \in \{1, 2, \ldots, n\}$. This implies that $\| \sum_{k=1}^n a_k e_k\|_1 \geq |a_m| > 0$. Moreover, if $\sum_{k=1}^n a_k e_k \in \mathcal{X}$ and $b \in K$ then

$$\left\| b \sum_{k=1}^n a_k e_k \right\|_1 = \sum_{k=1}^n |ba_k| = |b| \left\| \sum_{k=1}^n a_k e_k \right\|_1 .$$

If $\sum_{k=1}^n a_k e_k, \sum_{k=1}^n b_k e_k \in \mathcal{X}$ then

$$\left\| \sum_{k=1}^n a_k e_k + \sum_{k=1}^n b_k e_k \right\|_1 = \sum_{k=1}^n |a_k + b_k| \leq \sum_{k=1}^n |a_k| + \sum_{k=1}^n |b_k| = \left\| \sum_{k=1}^n a_k e_k \right\|_1 + \left\| \sum_{k=1}^n b_k e_k \right\|_1 .$$

This shows that $\| \cdot \|_1$ is a norm on $\mathcal{X}$.

(b) Let $(a_1, a_2, \ldots, a_n) \in K^n$ and suppose that $\|(a_1, a_2, \ldots, a_n)\|_2 = 1$. Then $\sum_{k=1}^n |a_k|^2 = 1$, and in particular $|a_k|^2 \leq 1$ for all $k \in \{1, 2, \ldots, n\}$. It follows that $|a_k| \leq 1$ for all $k \in \{1, 2, \ldots, n\}$, in which case

$$\left\| \sum_{k=1}^n a_k e_k \right\|_1 = \sum_{k=1}^n |a_k| \leq n,$$

and hence the linear map $(a_1, a_2, \ldots, a_n) \mapsto \sum_{k=1}^n a_k e_k$ is bounded, thus continuous.

(c) Define $C \coloneqq \{(a_1, a_2, \ldots, a_n) \in K^n \mid \| \sum_{k=1}^n a_k e_k\|_1 = 1\}$, so that the image of C under the map from (b) is the given set. Hence, it suffices to show that C is compact. Note that $\| \cdot \|_1 : \mathcal{X} \to \mathbb{R}$ is (Lipschitz) continuous, so the map $(a_1, a_2, \ldots, a_n) \mapsto \| \sum_{k=1}^n a_k e_k\|_1$ from K^n to $\mathbb{R}$ is continuous, implying that C (the preimage of $\{1\}$) is closed. Moreover C is bounded, thus compact, because for each $(a_1, a_2, \ldots, a_n) \in K^n$

$$\|(a_1, a_2, \ldots, a_n)\|_2 \leq \sum_{k=1}^n \|(0, \ldots, 0, a_k, 0, \ldots, 0)\|_2 = \sum_{k=1}^n |a_k| = 1.$$

(d) Let $\| \cdot \| : \mathcal{X} \to \mathbb{R}$ be a norm on $\mathcal{X}$. If $\sum_{k=1}^n a_k e_k \in \mathcal{X}$ then

$$\left\| \sum_{k=1}^n a_k e_k \right\| \leq \sum_{k=1}^n |a_k| \|e_k\| \leq \max\{\|e_k\|\}_{k=1}^n \sum_{k=1}^n |a_k| = \max\{\|e_k\|\}_{k=1}^n \left\| \sum_{k=1}^n a_k e_k \right\|_1 .$$

This proves one half of the equivalence and shows that $\| \cdot \|$ is (Lipschitz) continuous on $(\mathcal{X}, \| \cdot \|_1)$. It follows that $\| \cdot \|$ has a minimum at some $u \in B$, where B is the compact set $\{x \in \mathcal{X} \mid \|x\|_1 = 1\}$. Since $0 \notin B$, $u \neq 0$ and hence $\|u\| > 0$. If $x \in \mathcal{X}$ is non-zero then

$$\|x\| = \|x\|_1 \left\| \frac{x}{\|x\|_1} \right\| \geq \|x\|_1 \|u\|,$$

which also holds if $x = 0$. This proves the other half of the equivalence, so every norm on $\mathcal{X}$ is equivalent to $\| \cdot \|_1$.

7. (a) Since $\mathcal{X}$ is a Banach space, $L(\mathcal{X}, \mathcal{X})$ is also a Banach space. The series $\sum_{k=0}^\infty (I - T)^k$ is absolutely convergent because $\sum_{k=0}^\infty \|(I - T)^k\| \leq \sum_{k=0}^\infty \|I - T\|^k$, which is a geometric series with ratio $\|I - T\| < 1$. Therefore

$\sum_{k=0}^{\infty}(I-T)^k$ converges to some $S \in L(\mathcal{X},\mathcal{X})$. Given $\varepsilon \in (0,\infty)$, there exists $N \in \mathbb{N}$ such that $\|S - \sum_{k=0}^{n}(I-T)^k\| < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq N$, in which case

$$\begin{aligned}
\|S - I - S(I-T)\| &= \left\| S - \sum_{k=0}^{N+1}(I-T)^k + \sum_{k=1}^{N+1}(I-T)^k - S(I-T) \right\| \\
&\leq \left\| S - \sum_{k=0}^{N+1}(I-T)^k \right\| + \left\| \sum_{k=1}^{N+1}(I-T)^{k-1}(I-T) - S(I-T) \right\| \\
&< \varepsilon + \left\| \left(\sum_{k=0}^{N}(I-T)^k - S \right)(I-T) \right\| \\
&\leq \varepsilon + \left\| \sum_{k=0}^{N}(I-T)^k - S \right\| \|I-T\| \\
&< \varepsilon + \varepsilon\|I-T\| \\
&< 2\varepsilon.
\end{aligned}$$

This implies that $\|S - I - S(I-T)\| = 0$, so $S - I = S(I-T) = S - ST$ and hence $ST = I$. A very similar calculation shows that $\|S - I - (I-T)S\| = 0$ and hence $TS = I$.

(b) Since $\|I - T^{-1}S\| = \|T^{-1}S - I\| = \|T^{-1}S - T^{-1}T\| \leq \|T^{-1}\|\|S-T\| < \|T^{-1}\|\|T^{-1}\|^{-1} = 1$, exercise (a) implies that $T^{-1}S$ has an inverse $R \in L(\mathcal{X},\mathcal{X})$. Then RT^{-1} is an inverse for S, as $(RT^{-1})S = R(T^{-1}S) = I$ and $S(RT^{-1}) = T(T^{-1}S)RT^{-1} = TT^{-1} = I$. Thus every element of $B_{\|T^{-1}\|^{-1}}(T)$ is invertible for all invertible $T \in L(\mathcal{X},\mathcal{X})$, so the set of invertible elements of $L(\mathcal{X},\mathcal{X})$ is open.

8. Clearly $M(X)$ is a $\mathbb{C}$-space. For each $\nu \in M(X)$ there is a finite measure μ on X and a function $f \in L^1(\mu)$ such that $d\nu = f\,d\mu$. Note that $\|\nu\| = |\nu|(X) = \int |f|\,d\nu$ is zero iff f vanishes almost everywhere, or equivalently $\nu = 0$. If $a \in \mathbb{C}$ then $\|a\nu\| = |a\nu|(X) = \int |af|\,d\mu = |a|\int |f|\,d\mu = |a||\nu|(X) = |a|\|\nu\|$ because $d(a\nu) = (af)\,d\mu$. The triangle inequality follows from Proposition 3.14. Therefore $\|\cdot\|$ is a norm on $M(X)$.

If $\sum_{n=1}^{\infty}\nu_n$ is an absolutely convergent series in $M(X)$, and $A \in \mathcal{M}$, then $\sum_{n=1}^{\infty}\nu_n(A)$ also converges absolutely because $|\nu_n(A)| \leq |\nu_n|(A) \leq |\nu_n|(X)$ for all $n \in \mathbb{N}$. Hence, we may define $\nu : \mathcal{M} \to \mathbb{C}$ by $\nu(A) := \sum_{n=1}^{\infty}\nu_n(A)$. Clearly $\nu(\varnothing) = 0$; if $(A_k)_{k=1}^{\infty}$ is a sequence of disjoint elements of $\mathcal{M}$ whose union is $A \in \mathcal{M}$ then

$$\sum_{k=1}^{\infty}|\nu(A_k)| \leq \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|\nu_n(A_k)| = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\nu_n(A_k)| \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\nu_n|(A_k) = \sum_{n=1}^{\infty}|\nu_n|(A) \leq \sum_{n=1}^{\infty}\|\nu_n\| < \infty$$

by Tonelli's theorem, and Fubini's theorem implies that

$$\sum_{k=1}^{\infty}\nu(A_k) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\nu_n(A_k) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\nu_n(A_k) = \sum_{n=1}^{\infty}\nu_n(A) = \nu(A).$$

In other words $\sum_{k=1}^{\infty}\nu(A_k)$ converges absolutely to $\nu(A)$, which shows that $\nu \in M(X)$. It remains to show that $\sum_{n=1}^{\infty}\nu_n$ converges to ν in $M(X)$. If $N, M \in \mathbb{N}$ and $E_1,\ldots,E_M \in \mathcal{M}$ are disjoint sets covering X, then

$$\sum_{m=1}^{M}\left|\left(\nu - \sum_{n=1}^{N-1}\nu_n\right)(E_m)\right| = \sum_{m=1}^{M}\left|\sum_{n=N}^{\infty}\nu_n(E_m)\right| \leq \sum_{m=1}^{M}\sum_{n=N}^{\infty}|\nu_n|(E_m) = \sum_{n=N}^{\infty}\sum_{m=1}^{M}|\nu_n|(E_m) = \sum_{n=N}^{\infty}\|\nu_n\|.$$

Therefore $\|\nu - \sum_{n=1}^{N-1}\nu_n\| \leq \sum_{n=N}^{\infty}\|\nu_n\|$ by Exercise 3.21 (Homework 6). Now take the limit $N \to \infty$.

9. (a) Let $f \in C([0,1])$ and suppose that f is k times continuously differentiable on $(0,1)$ with $\lim_{x\searrow 0} f^{(j)}(x)$ and $\lim_{x\nearrow 1} f^{(j)}(x)$ existing for all $j \in \{0,1,\ldots,k\}$. Fix $j \in \{1,2,\ldots,k\}$ and assume (using induction) that $f^{(j-1)}$ is continuous on $[0,1]$. Let $\varepsilon \in (0,\infty)$ and define $m := \lim_{x\searrow 0} f^{(j)}(x)$. Then there exists $\delta \in (0,\infty)$ such that $|\operatorname{Re}(f^{(j)}(x)) - \operatorname{Re}(m)| < \varepsilon$ for all $x \in (0,\delta)$. Now let $x \in (0,\delta)$. By the mean value theorem there exists $x_0 \in (0,x)$ such that $\operatorname{Re}(f^{(j)}(x_0))x = \operatorname{Re}(f^{(j)}(x_0))(x-0) = \operatorname{Re}(f^{(j-1)}(x)) - \operatorname{Re}(f^{(j-1)}(0))$. Thus

$$\left|\frac{\operatorname{Re}(f^{(j-1)}(x)) - \operatorname{Re}(f^{(j-1)}(0))}{x} - \operatorname{Re}(m)\right| = \left|\frac{\operatorname{Re}(f^{(j)}(x_0))x}{x} - \operatorname{Re}(m)\right| = |\operatorname{Re}(f^{(j)}(x_0)) - \operatorname{Re}(m)| < \varepsilon$$

because $x_0 \in (0,\delta)$. This shows that $\operatorname{Re}(f^{(j-1)})$ has a one-sided derivative at 0, since

$$\lim_{x\searrow 0} \frac{\operatorname{Re}(f^{(j-1)}(x)) - \operatorname{Re}(f^{(j-1)}(0))}{x} = \operatorname{Re}(m).$$

Similarly $\operatorname{Im}(f^{(j-1)})$ has one-sided derivative $\operatorname{Im}(m)$ at 0, so $f^{(j)}(0) = m$. Therefore $f^{(j)}$ is defined and continuous on $[0,1)$. A similar argument shows that $f^{(j)}(1) = \lim_{x\nearrow 1} f^{(j)}(x)$, so by induction $f \in C^k([0,1])$.

The converse is clear because $f^{(j)}$ is continuous for all $f \in C^k([0,1])$ and $j \in \{0,1,\ldots,k\}$.

(b) Clearly $\|f\| \geq 0$ for all $f \in C^k([0,1])$. If $f \in C^k([0,1])$ is non-zero then $\left\|f^{(0)}\right\|_u > 0$ and hence $\|f\| > 0$. Also, if $f \in C^k([0,1])$ and $a \in \mathbb{C}$ then

$$\|af\| = \sum_{j=0}^{k} \left\|(af)^{(j)}\right\|_u = \sum_{j=0}^{k} \left\|af^{(j)}\right\|_u = \sum_{j=0}^{k} |a| \left\|f^{(j)}\right\|_u = |a|\|f\|.$$

Moreover, if $f, g \in C^k([0,1])$ then

$$\|f+g\| = \sum_{j=0}^{k} \left\|(f+g)^{(j)}\right\|_u = \sum_{j=0}^{k} \left\|f^{(j)} + g^{(j)}\right\|_u \leq \sum_{j=0}^{k} \left\|f^{(j)}\right\|_u + \sum_{j=0}^{k} \left\|g^{(j)}\right\|_u = \|f\| + \|g\|.$$

This shows that $\|\cdot\|$ is a norm on $C^k([0,1])$. We assume (using induction) that $C^{k-1}([0,1])$ is complete under the corresponding norm (this is known for $k=1$ because $C^0([0,1]) = C([0,1])$). Let $\langle f_n\rangle_{n=1}^\infty$ be a Cauchy sequence in $C^k([0,1])$. For each $\varepsilon \in (0,\infty)$ there exists $N \in \mathbb{N}$ such that

$$\sum_{j=0}^{k-1} \left\|f_n^{(j)} - f_m^{(j)}\right\|_u \leq \sum_{j=0}^{k} \left\|f_n^{(j)} - f_m^{(j)}\right\|_u = \|f_n - f_m\| < \varepsilon$$

for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$. This shows that $\langle f_n\rangle_{n=1}^\infty$ is Cauchy in $C^{k-1}([0,1])$, so it converges to some $f \in C^{k-1}([0,1])$. Similarly $\left\langle f_n^{(k)}\right\rangle_{n=1}^\infty$ is Cauchy in $C([0,1])$, so it converges to some $g \in C([0,1])$. Note that $\left\langle f_n^{(k-1)}\right\rangle_{n=1}^\infty$ converges in $C([0,1])$ to $f^{(k-1)}$, because $\left\|f_n^{(k-1)} - f^{(k-1)}\right\|_u \leq \sum_{j=0}^{k-1} \left\|f_n^{(j)} - f^{(j)}\right\|_u$ for all $n \in \mathbb{N}$. Since $|g|$ is bounded above by a constant function, $\left\langle \left|f_n^{(k)}\right|\right\rangle_{n=1}^\infty$ is eventually dominated by a slightly larger constant function (in $L^1([0,1])$). By the dominated convergence theorem, it follows that

$$f^{(k-1)}(x) - f^{(k-1)}(0) = \lim_{n\to\infty}\left(f_n^{(k-1)}(x) - f_n^{(k-1)}(0)\right) = \lim_{n\to\infty}\int_0^x f_n^{(k)} = \int_0^x \lim_{n\to\infty} f_n^{(k)} = \int_0^x g$$

for all $x \in [0,1]$, and hence $f^{(k)} = g$ by exercise (a). This implies that $f \in C^k([0,1])$. Moreover $\langle f_n\rangle_{n=1}^\infty$ converges to f in $C^k([0,1])$, because $\|f_n - f\| = \sum_{j=0}^{k-1}\left\|f_n^{(j)} - f^{(j)}\right\|_u + \left\|f_n^{(k)} - g\right\|_u$ for all $n \in \mathbb{N}$. This shows (by induction) that $C^k([0,1])$ is a Banach space for all $k \in \mathbb{N}$.

10. Let $\langle f_n\rangle_{n=1}^\infty$ be a Cauchy sequence in $L^1_k([0,1])$. If $j\in\{0,1,\dots,k\}$ then $\int_0^1\left|f_n^{(j)}-f_m^{(j)}\right|\le\|f_n-f_m\|$ for all $m,n\in\mathbb{N}$, so $\left\langle f_n^{(j)}\right\rangle_{n=1}^\infty$ is (modulo equality almost everywhere) a Cauchy sequence in the Banach space $L^1([0,1])$, so it has a limit $g_j\in L^1([0,1])$. Fix $j\in\{0,1,\dots,k-1\}$, $n\in\mathbb{N}$, and note that $f_n^{(j)}(x)-f_n^{(j)}(0)=\int_0^x f_n^{(j+1)}$ for all $x\in[0,1]$, either because $f_n^{(j)}$ is absolutely continuous or $f_n^{(j+1)}$ is continuous. Choose a subsequence $\left\langle f_{n_i}^{(j)}\right\rangle_{i=1}^\infty$ which converges to g_j pointwise almost everywhere, and note that $\left\langle f_{n_i}^{(j+1)}\right\rangle_{i=1}^\infty$ converges in $L^1([0,1])$ to g_{j+1}. Taking a further subsequence, we may assume that $\left\langle f_{n_i}^{(j+1)}\right\rangle_{i=1}^\infty$ converges to g_{j+1} pointwise almost everywhere. This implies that $\left\langle\left|f_{n_i}^{(j+1)}-g_{j+1}\right|+|g_{j+1}|\right\rangle_{i=1}^\infty$ converges to $|g_{j+1}|$ pointwise almost everywhere, and clearly

$$\lim_{i\to\infty}\int_0^1\left(\left|f_{n_i}^{(j+1)}-g_{j+1}\right|+|g_{j+1}|\right)=0+\lim_{i\to\infty}\int_0^1|g_{j+1}|=\int_0^1|g_{j+1}|,$$

while $\left|f_{n_i}^{(j+1)}\right|\le\left|f_{n_i}^{(j+1)}-g_{j+1}\right|+|g_{j+1}|$ for all $i\in\mathbb{N}$. Hence, by the generalised dominated convergence theorem

$$g_j(x)-g_j(0)=\lim_{i\to\infty}\left(f_{n_i}^{(j)}(x)-f_{n_i}^{(j)}(0)\right)=\lim_{i\to\infty}\int_0^x f_{n_i}^{(j+1)}=\int_0^x g_{j+1}$$

for almost all $x\in[0,1]$. This shows that g_j is almost everywhere equal to the absolutely continuous function $h_j:[0,1]\to\mathbb{C}$ defined by $h_j(x):=g_j(0)+\int_0^x g_{j+1}$. If $j<k-1$, it follows that $h_j(x)-h_j(0)=\int_0^x h_{j+1}$ for all $x\in[0,1]$, so that $h_j'=h_{j+1}$, by the fundamental theorem of calculus and exercise 9(a). By induction $h_0^{(j)}=h_j$ for all $j\in\{0,1,\dots,k-1\}$. Since $h_0^{(k-1)}=h_{k-1}$ is absolutely continuous, it follows that $h_0\in L^1_k([0,1])$ and $\int_0^x h_0^{(k)}=h_0^{(k-1)}(x)-h_0^{(k-1)}(0)=\int_0^x g_k$ for all $x\in[0,1]$. The Borel measures $E\mapsto\int_E h_0^{(k)}$ and $E\mapsto\int_E g_k$ agree on intervals, so they are equal and hence $h_0^{(k)}=g_k$ almost everywhere. Given $\varepsilon\in(0,\infty)$ and $j\in\{0,1,\dots,k\}$, there exists $N_j\in\mathbb{N}$ such that $\int_0^1\left|f_n^{(j)}-h_0^{(j)}\right|=\int_0^1\left|f_n^{(j)}-g_j\right|<\frac{\varepsilon}{k+1}$ for all $n\in\mathbb{N}$ with $n\ge N_j$. It follows that $\|f_n-h_0\|<\varepsilon$ for all $n\in\mathbb{N}$ with $n\ge\max\{N_j\}_{j=0}^k$. so $\langle f_n\rangle_{n=1}^\infty$ converges to h_0 in $L^1_k([0,1])$. Therefore $L^1_k([0,1])$ is a Banach space.

11. (a) Clearly $\|f\|_{\Lambda_\alpha}\ge0$ for all $f\in\Lambda_\alpha([0,1])$. If $f\in\Lambda_\alpha([0,1])$ is non-zero, then there exists $x\in[0,1]$ such that $f(x)\ne0$, and either $f(x)=f(0)$ or $x\ne0$ and $\frac{|f(x)-f(0)|}{|x-0|^\alpha}>0$, in which case $\|f\|_{\Lambda_\alpha}>0$. Moreover,

$$\begin{aligned}\|af\|_{\Lambda_\alpha}&=|af(0)|+\sup_{x,y\in[0,1],\ x\ne y}\frac{|af(x)-af(y)|}{|x-y|^\alpha}\\&=|a||f(0)|+\sup_{x,y\in[0,1],\ x\ne y}|a|\frac{|f(x)-f(y)|}{|x-y|^\alpha}\\&=|a|\left(|f(0)|+\sup_{x,y\in[0,1],\ x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right)\\&=|a|\|f\|_{\Lambda_\alpha}\end{aligned}$$

for all $a\in\mathbb{C}$ and $f\in\Lambda_\alpha([0,1])$. Finally, if $f,g\in\Lambda_\alpha([0,1])$ then

$$\begin{aligned}\frac{|f(x')+g(x')-f(y')-g(y')|}{|x'-y'|^\alpha}&\le\frac{|f(x')-f(y')|}{|x'-y'|^\alpha}+\frac{|g(x')-g(y')|}{|x'-y'|^\alpha}\\&\le\sup_{x,y\in[0,1],\ x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}+\sup_{x,y\in[0,1],\ x\ne y}\frac{|g(x)-g(y)|}{|x-y|^\alpha}\end{aligned}$$

for all $x',y'\in[0,1]$ with $x'\ne y'$, and hence

$$\|f+g\|_{\Lambda_\alpha}=|f(0)+g(0)|+\sup_{x,y\in[0,1],\ x\ne y}\frac{|f(x)+g(x)-f(y)-g(y)|}{|x-y|^\alpha}$$

$$\leq |f(0)| + |g(0)| + \sup_{x,y\in[0,1],\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} + \sup_{x,y\in[0,1],\ x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\alpha}$$
$$= \|f\|_{\Lambda_\alpha} + \|g\|_{\Lambda_\alpha}.$$

This shows that $\|\cdot\|_{\Lambda_\alpha}$ is a norm on $\Lambda_\alpha([0,1])$. Given $f \in \Lambda_\alpha([0,1])$ and $x \in (0,1]$, it is clear that

$$|f(x)| \leq |f(x)-f(0)| + |f(0)| = |f(0)| + \frac{|f(x)-f(0)|}{|x-0|^\alpha}|x|^\alpha \leq |f(0)| + \frac{|f(x)-f(0)|}{|x-0|^\alpha} \leq \|f\|_{\Lambda_\alpha},$$

which implies that $\|f\|_u \leq \|f\|_{\Lambda_\alpha}$ because $|f(0)| \leq \|f\|_{\Lambda_\alpha}$ by definition. It follows that every sequence $\langle f_n\rangle_{n=1}^\infty$ which is Cauchy in $\Lambda_\alpha([0,1])$ is also Cauchy in $B([0,1])$, and therefore has a limit $f \in B([0,1])$. Given $x, y \in [0,1]$ with $x \neq y$, there exists $n \in \mathbb{N}$ such that $\|f - f_n\|_u < |x-y|^\alpha$, and hence

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq \frac{|f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)|}{|x-y|^\alpha} < 1 + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} + 1 \leq \|f_n\|_{\Lambda_\alpha} + 2.$$

This implies that $f \in \Lambda_\alpha([0,1])$ because every Cauchy sequence is bounded (in $\Lambda_\alpha([0,1])$). Given $\varepsilon \in (0,\infty)$ there exists $N \in \mathbb{N}$ such that $\|f_m - f_n\|_{\Lambda_\alpha} < \frac{\varepsilon}{4}$ for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$. If $n \in \mathbb{N}$ and $x, y \in [0,1]$ with $n \geq N$ and $x \neq y$, there exists $m \in \mathbb{N}$ such that $m \geq n$ and $\|f - f_m\|_u < \frac{\varepsilon}{6}|x-y|^\alpha \leq \frac{\varepsilon}{6}$, so

$$\begin{aligned}
\frac{|f(x)-f_n(x)-f(y)+f_n(y)|}{|x-y|^\alpha} &= \frac{|f(x)-f_m(x)+f_m(x)-f_n(x)+f_n(y)-f_m(y)+f_m(y)-f(y)|}{|x-y|^\alpha} \\
&\leq \frac{|f(x)-f_m(x)| + |f_m(x)-f_n(x)+f_n(y)-f_m(y)| + |f_m(y)-f(y)|}{|x-y|^\alpha} \\
&< \frac{\varepsilon}{6} + \|f_m - f_n\|_{\Lambda_\alpha} + \frac{\varepsilon}{6} \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{3}
\end{aligned}$$

and hence

$$\begin{aligned}
\|f - f_n\|_{\Lambda_\alpha} &= |f(0)-f_n(0)| + \sup_{x,y\in[0,1],\ x\neq y} \frac{|f(x)-f_n(x)-f(y)+f_n(y)|}{|x-y|^\alpha} \\
&\leq |f(0)-f_m(0)| + |f_m(0)-f_n(0)| + \frac{\varepsilon}{4} + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{6} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Therefore $\langle f_n\rangle_{n=1}^\infty$ converges to f in $\Lambda_\alpha([0,1])$, which shows that $\Lambda_\alpha([0,1])$ is a Banach space.

(b) Clearly $\lambda_1([0,1])$ consists of all differentiable functions $f : [0,1] \to \mathbb{C}$ such that $f' = 0$. By the mean value theorem these functions have constant real and imaginary parts, so $\lambda_1([0,1])$ consists only of constant functions. Now suppose that $\alpha \in (0,1)$. Clearly $0 \in \lambda_\alpha([0,1])$. If $f \in \lambda_\alpha([0,1])$ and $a \in \mathbb{C}$ then

$$\lim_{x\to y} \frac{|af(x)-af(y)|}{|x-y|} = \lim_{x\to y} |a| \frac{|f(x)-f(y)|}{|x-y|} = |a| \cdot 0 = 0$$

for all $y \in [0,1]$, so $af \in \lambda_\alpha([0,1])$. Given $f, g \in \lambda_\alpha([0,1])$ and $x, y \in [0,1]$ with $x \neq y$,

$$0 \leq \frac{|f(x)+g(x)-f(y)-g(y)|}{|x-y|} \leq \frac{|f(x)-f(y)|}{|x-y|} + \frac{|g(x)-g(y)|}{|x-y|}$$

so $f+g \in \lambda_\alpha([0,1])$ by the squeeze theorem. This shows that $\lambda_\alpha([0,1])$ is a subspace of $\Lambda_\alpha([0,1])$. Let $\langle f_n \rangle_{n=1}^\infty$ be a sequence in $\lambda_\alpha([0,1])$ which converges to $f \in \Lambda_\alpha([0,1])$. Given $\varepsilon \in (0,\infty)$ and $y \in [0,1]$, there exists $n \in \mathbb{N}$ such that $\|f - f_n\|_{\Lambda_\alpha} < \frac{\varepsilon}{2}$ and $\delta \in (0,\infty)$ such that

$$\frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha} < \frac{\varepsilon}{2}$$

for all $x \in [0,1]$ with $0 < |x-y| < \delta$. It follows that

$$\frac{|f(x) - f(y)|}{|x-y|^\alpha} \le \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|^\alpha} + \frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha} < \|f - f_n\|_{\Lambda_\alpha} + \frac{\varepsilon}{2} < \varepsilon,$$

which shows that $f \in \lambda_\alpha([0,1])$. Therefore $\lambda_\alpha([0,1])$ is a closed subspace of $\Lambda_\alpha([0,1])$. For each $\beta \in (\alpha, 1)$, define $f_\beta : [0,1] \to \mathbb{R}$ by $f_\beta(x) := x^\beta$. Then $f_\beta'(x) = \beta x^{\beta-1}$ and hence $f_\beta''(x) = \beta(\beta-1)x^{\beta-2}$ for all $x \in (0,1]$. This implies that $f_\beta'' \le 0$ on $(0,1]$, so f_β' is decreasing on $(0,1]$ by the mean value theorem. Given $x, y \in (0,1]$ with $x < y$, there exist $a \in (0,x)$, $b \in (x,y)$, $c \in (0, y-x)$ and $d \in (y-x, y)$ such that

$$\frac{f_\beta(x)}{x} = \frac{f_\beta(x) - f_\beta(0)}{x - 0} = f_\beta'(a) \ge f_\beta'(b) = \frac{f_\beta(y) - f_\beta(x)}{y - x}$$

and

$$\frac{f_\beta(y-x)}{y-x} = \frac{f_\beta(y-x) - f_\beta(0)}{y-x-0} = f_\beta'(c) \ge f_\beta'(d) = \frac{f_\beta(y) - f_\beta(y-x)}{y-(y-x)} = \frac{f_\beta(y) - f_\beta(y-x)}{x},$$

again by the mean value theorem. Therefore $f_\beta(x)(y-x) \ge (f_\beta(y) - f_\beta(x))x$, so that $f_\beta(x)y \ge f_\beta(y)x$, and $f_\beta(y-x)x \ge f_\beta(y)(y-x) - f_\beta(y-x)(y-x)$, whence $f_\beta(y-x)y \ge f_\beta(y)(y-x)$. These inequalities combine to give $f_\beta(y-x)y + f_\beta(x)y \ge f_\beta(y)y$, and hence $f_\beta(y-x) \ge f_\beta(y) - f_\beta(x)$. It follows that

$$0 \le \frac{|x^\beta - y^\beta|}{|x-y|^\alpha} = \frac{|x^\beta - y^\beta|}{|x-y|^\beta}|x-y|^{\beta-\alpha} = \frac{f_\beta(y) - f_\beta(x)}{f_\beta(y-x)}|x-y|^{\beta-\alpha} \le |x-y|^{\beta-\alpha},$$

which also holds if $x = 0$ and $y \in (0,1]$. This shows that $f \in \Lambda_\alpha([0,1])$ since $|x-y|^{\beta-\alpha} \le 2^{\beta-\alpha}$, and the squeeze theorem implies that $f \in \lambda_\alpha([0,1])$. It remains to show that $\{f_\beta\}_{\beta\in(\alpha,1)}$ is linearly independent. To this end, let $n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n \in \mathbb{C}$ and $\beta_1, \beta_2, \ldots, \beta_n \in (\alpha, 1)$. Suppose $a_1 f_{\beta_1} + a_2 f_{\beta_2} + \cdots + a_n f_{\beta_n} = 0$. Then $\sum_{i=1}^n a_i \prod_{j=0}^{k-1}(\beta_i - j) = a_1 f_{\beta_1}^{(k)}(1) + a_2 f_{\beta_2}^{(k)}(1) + \cdots + a_n f_{\beta_n}^{(k)}(1) = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Therefore

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j=0}^{n-2}(\beta_1 - j) & \prod_{j=0}^{n-2}(\beta_1 - j) & \cdots & \prod_{j=0}^{n-2}(\beta_1 - j) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which has non-trivial solutions iff the rows of the above $n \times n$ matrix are linearly dependent. If this were the case, some linear combination of the polynomials $\prod_{j=0}^{k-1}(x-j)$ would be a non-zero polynomial of degree $n-1$ with n distinct roots $\beta_1, \beta_2, \ldots, \beta_n$. This is impossible, so it must be the case that $a_1 = a_2 = \cdots = a_n = 0$.

12. (a) If $X \in \mathcal{X}/\mathcal{M}$ then $\|X\| = \inf_{y\in X}\|y\|$ is independent of the representation of X. It is clear that $\|\mathcal{M}\| = 0$. Conversely, if $x \in \mathcal{X}$ and $\|x + \mathcal{M}\| = 0$ then there is a sequence $(y_n)_{n=1}^\infty$ in $\mathcal{M}$ such that $\lim_{n\to\infty}\|x + y_n\| = 0$. It follows that $x = \lim_{n\to\infty}(-y_n) \in \mathcal{M}$, i.e. $x + \mathcal{M} = \mathcal{M}$. If $x \in \mathcal{X}$ and $a \in K^\times$ then

$$\|a(x+\mathcal{M})\| = \|ax + \mathcal{M}\| = \inf_{y\in\mathcal{M}}\|ax + y\| = |a|\inf_{y\in\mathcal{M}}\|x + a^{-1}y\| = |a|\|x + \mathcal{M}\|.$$

The case $a = 0$ follows from above. Finally, if $x, y \in \mathcal{X}$ then

$$\begin{aligned}\|(x+y)+\mathcal{M}\| &= \inf_{z\in\mathcal{M}} \|x+y+z\| \\ &= \inf_{w,z\in\mathcal{M}} \|x+y+w+z\| \\ &\leq \inf_{w,z\in\mathcal{M}} (\|x+y\|+\|w+z\|) \\ &= \inf_{w\in\mathcal{M}} \|x+w\| + \inf_{z\in\mathcal{M}} \|y+z\| \\ &= \|x+\mathcal{M}\|+\|y+\mathcal{M}\|.\end{aligned}$$

(b) Since $\mathcal{M} \neq \mathcal{X}$, there exists $x \in \mathcal{X} \setminus \mathcal{M}$. If $\varepsilon \geq 1$ we are done, otherwise $1 < \frac{1}{1-\varepsilon}$, so $\|x+\mathcal{M}\| < \frac{\|x+\mathcal{M}\|}{1-\varepsilon}$ and there exists $y \in \mathcal{M}$ such that $\|x-y\| < \frac{\|x+\mathcal{M}\|}{1-\varepsilon}$. The idea is that y is close to x, so $x-y$ is close to being "normal" to $\mathcal{M}$. Thus we define $z := \|x-y\|^{-1}(x-y)$, and note that $\|z\| = 1$ while

$$\|z+\mathcal{M}\| = \frac{\|(x-y)+\mathcal{M}\|}{\|x-y\|} = \frac{\|x+\mathcal{M}\|}{\|x-y\|} > 1-\varepsilon.$$

(c) Clearly $\|\pi(x)\| = \inf_{y\in\mathcal{M}} \|x+y\| \leq \|x\|$ for all $x \in \mathcal{X}$, which means (π is bounded and) $\|\pi\| \leq 1$. Conversely, if $\varepsilon \in (0,1)$ then $1-\varepsilon < \|\pi(x)\| \leq \|\pi\|\|x\|$ for some $x \in \mathcal{X}$ with $\|x\| = 1$. This implies that $1 \leq \|\pi\|$.

(d) Let $\sum_{n=1}^{\infty} X_n$ be an absolutely convergent series in $\mathcal{X}/\mathcal{M}$. For each $n \in \mathbb{N}$, note that $\|X_n\| = \inf_{x\in X_n} \|x\|$, and hence there exists $x_n \in X_n$ such that $\|x_n\| < \|X_n\| + 2^{-n}$. Clearly $\sum_{n=1}^{\infty} x_n$ is an absolutely convergent series in $\mathcal{X}$, so it has a limit $x \in \mathcal{X}$. Since π is bounded, $\sum_{n=1}^{\infty} X_n = \sum_{n=1}^{\infty} \pi(x_n) = \pi(\sum_{n=1}^{\infty} x_n) = \pi(x)$.

(e) If $U \subseteq \mathcal{X}/\mathcal{M}$ is norm-open, then $\pi^{-1}(U)$ is open and hence U is open in the quotient topology. Conversely, if $U \subseteq \mathcal{X}/\mathcal{M}$ is open in the quotient topology, then $\pi^{-1}(U)$ is open. Given $x+\mathcal{M} \in U$, there exists $r \in (0,\infty)$ such that $B_r(x) \subseteq \pi^{-1}(U)$. If $y+\mathcal{M} \in B_r(x+\mathcal{M})$ then $\|x-y+\mathcal{M}\| < r$, so there exists $z \in \mathcal{M}$ such that $\|x-y+z\| < r$. In particular $y-z \in \pi^{-1}(U)$, which implies that $y+\mathcal{M} = \pi(y-z) \in U$. Therefore $B_r(x+\mathcal{M}) \subseteq U$, which shows that U is norm-open.

13. Clearly $0 \in \mathcal{M}$. Moreover, if $a \in K$ and $x, y \in \mathcal{M}$ then $\|ax+y\| \leq |a|\|x\| + \|y| = 0$, so $ax+y \in \mathcal{M}$. Therefore $\mathcal{M}$ is a subspace of $\mathcal{X}$. If $x, y \in \mathcal{X}$ and $x+\mathcal{M} = y+\mathcal{M}$ then $\|x+\mathcal{M}\| = \|x\| \leq \|x-y\| + \|y\| = 0 + \|y+\mathcal{M}\|$, so $\|x+\mathcal{M}\| = \|y+\mathcal{M}\|$ by symmetry. Clearly $\|\mathcal{M}\| = 0$. Conversely, if $x \in \mathcal{X}$ and $\|x+\mathcal{M}\| = 0$ then $\|x\| = 0$, which means $x \in \mathcal{M}$ and hence $x+\mathcal{M} = \mathcal{M}$. If $x \in \mathcal{X}$ and $a \in K$ then $\|ax+\mathcal{M}\| = \|ax\| = |a|\|x\| = |a|\|x+\mathcal{M}\|$. Finally, if $x, y \in \mathcal{X}$ then $\|(x+y)+\mathcal{M}\| = \|x+y\| \leq \|x\|+\|y\| = \|x+\mathcal{M}\|+\|y+\mathcal{M}\|$.

18. (a) Let $\mathcal{M}$ be a proper closed subspace of $\mathcal{X}$ and let $x \in \mathcal{X} \setminus \mathcal{M}$. There exists $f \in \mathcal{X}^*$ such that $f(x) \neq 0$ and $f(M) = \{0\}$. Let $\langle u_n + a_n x\rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{M} + Kx$ that converges to $y \in \mathcal{X}$. Then

$$f(y) = \lim_{n\to\infty} f(u_n + a_n x) = \lim_{n\to\infty} a_n f(x),$$

since f is continuous, so $\langle a_n\rangle_{n=1}^{\infty}$ converges to $a := f(y)/f(x)$, which implies that $\langle a_n x\rangle_{n=1}^{\infty}$ converges to ax. Therefore $\langle u_n\rangle_{n=1}^{\infty} = \langle (u_n + a_n x) - a_n x\rangle_{n=1}^{\infty}$ converges to $y - ax$, which lies in $\mathcal{M}$ because $\mathcal{M}$ is closed. It follows that $y \in \mathcal{M} + Kx$, which shows that $\mathcal{M} + Kx$ is closed.

(b) Every finite-dimensional subspace of $\mathcal{X}$ is of the form $\sum_{k=1}^{n} Kx_k$ for some $n \in \mathbb{N} \cup \{0\}$ and $x_1, x_2, \ldots, x_n \in \mathcal{X}$, so the result follows from exercise (a) by induction, starting from $\mathcal{M} = \{0\}$. Alternatively, observe that every n-dimensional subspace $\mathcal{M}$ of $\mathcal{X}$ has an induced norm equivalent to the usual norm on K^n, so $\mathcal{M}$ is a Banach space and therefore closed in $\mathcal{X}$.

19. (a) Choose $x_1 \in \mathcal{X}$ with $\|x_1\| = 1$. Given $n \in \mathbb{N}$ we construct x_{n+1} inductively, assuming we already have $x_1, x_2, \dots, x_n$. Since $\mathcal{M} := \sum_{k=1}^n Kx_k$ is finite-dimensional, it is a proper closed subspace of $\mathcal{X}$. There exists $y \in \mathcal{X} \setminus \mathcal{M}$, and $r := \inf_{u\in\mathcal{M}} \|y-u\| > 0$ because otherwise we could find a sequence in $\mathcal{M}$ converging to y. Choose $u \in \mathcal{M}$ with $\|y-u\| < 2r$, and define $x_{n+1} := \frac{y-u}{\|y-u\|}$. Then $\|x_{n+1}\| = 1$, and for all $v \in \mathcal{M}$

$$\|x_{n+1} - v\| = \left\| \frac{y-u}{\|y-u\|} - v \right\| = \frac{1}{\|y-u\|} \Big\| y - u - \|y-u\|v \Big\| > \frac{1}{2r} r = \frac{1}{2}.$$

In particular $\|x_{n+1} - x_k\| \geq \frac{1}{2}$ for all $k \in \{1, 2, \dots, n\}$.

(b) Suppose that $0 \in \mathcal{X}$ had a compact neighbourhood C, and choose $r \in (0, \infty)$ such that $B_{2r}(0) \subseteq C$. Since scalar multiplication is continuous, $\frac{1}{r}C$ is compact, and contains all $x \in \mathcal{X}$ with $\|x\| = 1$. By exercise (a), we can construct a sequence in $\frac{1}{r}C$ with no convergent subsequence (every ball of radius $\frac{1}{4}$ contains at most one point from the sequence). This is a contradiction because $\frac{1}{r}$ is sequentially compact. Therefore 0 does not have a compact neighbourhood, so $\mathcal{X}$ is not locally compact.

20. Let $x_1, \dots, x_m \in \mathcal{M}$ form a basis, and $T : K^m \to \mathcal{M}$ the corresponding isomorphism (of K-spaces). Clearly $\|T(\cdot)\|$ is a norm on K^m, and it is equivalent to $\|\cdot\|_1$ by Exercise 6. In other words T and T^{-1} are bounded. The projections $\pi_1, \dots, \pi_m : K^m \to K$ induce linear functionals $f_1, \dots, f_m : \mathcal{M} \to K$ such that

$$|f_i(x)| = |\pi_i(T^{-1}(x))| \leq \|T^{-1}(x)\|_1 \leq \|T^{-1}\|\|x\|$$

for all $i \in \{1, \dots, m\}$ and $x \in \mathcal{M}$. By Hahn-Banach, there exist linear functionals $F_1, \dots, F_m : \mathcal{X} \to K$ extending $f_1, \dots, f_m$ such that each is bounded with norm at most $\|T^{-1}\|$. Let $\mathcal{N}$ be the closed subspace $\cap_{i=1}^m \ker(F_i)$. Note that $\mathcal{M} \cap \mathcal{N} = \cap_{i=1}^m \ker(f_i) = T(\cap_{i=1}^m \ker(\pi_i)) = T(\{0\}) = \{0\}$. Moreover, if $x \in \mathcal{X}$ then $\sum_{i=1}^m F_i(x)x_i \in \mathcal{M}$ and $x - \sum_{i=1}^m F_i(x)x_i \in \mathcal{N}$, because $F_j(x - \sum_{i=1}^m F_i(x)x_i) = 0$ for all $j \in \{1, \dots, m\}$. This shows that $\mathcal{X} = \mathcal{M} + \mathcal{N}$.

22. (a) By definition $T^\dagger$ is linear. If $f \in \mathcal{Y}^*$ and $\|f\| = 1$ then

$$\|T^\dagger f\| = \|f \circ T\| \leq \|f\|\|T\| = \|T\|,$$

so that $T^\dagger \in L(\mathcal{Y}^*, \mathcal{X}^*)$ with

$$\|T^\dagger\| = \sup\{\|T^\dagger f\| \mid f \in \mathcal{Y}^*, \|f\| = 1\} \leq \|T\|.$$

If $\|T\| = 0$ it follows that $\|T^\dagger\| = \|T\|$. Otherwise, given $\varepsilon \in (0, \|T\|)$ there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|Tx\| > \|T\| - \varepsilon$. By a corollary of Hahn-Banach there exists $f \in \mathcal{X}^*$ such that $\|f\| = 1$ and $f(Tx) = \|Tx\|$. Therefore $|(T^\dagger f)(x)| = |f(Tx)| = |\|Tx\|| > \|T\| - \varepsilon$, in which case $\|T^\dagger f\| > \|T\| - \varepsilon$ and hence $\|T^\dagger\| > \|T\| - \varepsilon$. This implies that $\|T^\dagger\| \geq \|T\|$, so $\|T^\dagger\| = \|T\|$.

(b) The image of $x \in \mathcal{X}$ in $\mathcal{X}^{**}$ is the map $\widehat{x} : f \mapsto f(x)$. Therefore $T^{\dagger\dagger}(\widehat{x})(f) = \widehat{x}(T^\dagger f) = (T^\dagger f)(x) = f(Tx)$ for all $f \in \mathcal{Y}^*$. But $\widehat{Tx} : f \mapsto f(Tx)$, which implies that $T^{\dagger\dagger}(\widehat{x}) = \widehat{Tx}$. Therefore $T^{\dagger\dagger}|_{\widehat{\mathcal{X}}} = T$ modulo the identifications $\mathcal{X} = \widehat{\mathcal{X}}$ and $\mathcal{Y} = \widehat{\mathcal{Y}}$.

(c) Since $T(\mathcal{X})$ is a subspace of $\mathcal{Y}$, so is $\overline{T(\mathcal{X})}$. Suppose $T(\mathcal{X})$ is not dense in $\mathcal{Y}$. Then there exists $y \in \mathcal{Y} \setminus \overline{T(\mathcal{X})}$, and hence $f \in \mathcal{Y}^*$ such that $f(y) \neq 0$ but $f(\overline{T(\mathcal{X})}) = \{0\}$. It follows that $T^\dagger f = f \circ T = 0$, so f is a non-zero element of $\ker(T^\dagger)$, which shows that $T^\dagger$ is not injective.

Conversely, suppose that $T^\dagger$ is not injective. Then there exists $f \in \ker(T^\dagger)$ such that $f \neq 0$. Hence, there exists $y \in \mathcal{Y}$ such that $f(y) \neq 0$. Since f is continuous, y has a neighbourhood U on which $|f| > 0$. Since $f \circ T = T^\dagger f = 0$, it follows that U is disjoint from $T(\mathcal{X})$. Therefore $\overline{T(\mathcal{X})} \subseteq U^c$, so $T(\mathcal{X})$ is not dense in $\mathcal{Y}$.

(d) Suppose T is not injective. Then $Tx = 0$ for some non-zero $x \in \mathcal{X}$. Since T is linear, we may assume $\|x\| = 1$. There exists $f \in \mathcal{X}^*$ such that $\|f\| = 1$ and $f(x) = \|x\| = 1$. If $g \in \mathcal{X}^*$ and $\|f - g\| < 1$ then

$$|1 - g(x)| = |f(x) - g(x)| = |(f - g)(x)| \leq \|f - g\|\|x\| < 1,$$

which implies that $g(x) \neq 0$. Since $g(x) = 0$ for all $g \in T^\dagger(\mathcal{Y}^*) = \{h \circ T \mid h \in \mathcal{Y}^*\}$, it follows that $B_1(f)$ is disjoint from $T^\dagger(\mathcal{Y}^*)$. Therefore $\overline{T^\dagger(\mathcal{Y}^*)} \subseteq B_1(f)^c$, so $T^\dagger(\mathcal{Y}^*)$ is not dense in $\mathcal{X}^*$.

Now suppose that $\mathcal{X}$ is reflexive and that $T^\dagger(\mathcal{Y}^*)$ is not dense in $\mathcal{X}^*$. Then there exists $f \in \mathcal{X}^* \setminus \overline{T^\dagger(\mathcal{Y}^*)}$, and hence $\widehat{x} \in \mathcal{X}^{**}$ such that $\widehat{x}(f) \neq 0$ but $\widehat{x}(\overline{T^\dagger(\mathcal{Y}^*)}) = \{0\}$. As the name suggests, $\widehat{x}$ corresponds to some $x \in \mathcal{X}$. Therefore $f(x) \neq 0$ but $g(x) = 0$ for all $g \in \overline{T^\dagger(\mathcal{Y}^*)}$. In particular $x \neq 0$. If $Tx \neq 0$ then there exists $g \in \mathcal{Y}^*$ such that $g(Tx) \neq 0$. This is a contradiction because $g \circ T = T^\dagger g \in T^\dagger(\mathcal{Y}^*)$. Therefore x is a non-zero element of $\ker(T)$, in which case T is not injective.

23. (a) $\mathcal{M}^0 = \cap_{x \in \mathcal{M}} \ker(\widehat{x})$ and $\mathcal{N}^\perp = \cap_{f \in \mathcal{N}} \ker(f)$.

(b) If $x \in \mathcal{M}$, then $f(x) = 0$ for all $f \in \mathcal{M}^0$, so $x \in (\mathcal{M}^0)^\perp$. Conversely, given $x \in \mathcal{X} \setminus \mathcal{M}$ there exists (by Hahn-Banach) $f \in \mathcal{X}^*$ such that $f|_\mathcal{M} = 0$ (i.e. $f \in \mathcal{M}^0$) but $f(x) \neq 0$, so $x \notin (\mathcal{M}^0)^\perp$.

If $f \in \mathcal{N}$ then $f(x) = 0$ for all $x \in \mathcal{N}^\perp$, so $f \in (\mathcal{N}^\perp)^0$. Conversely, given $f \notin \mathcal{N}$ there exists (by Hahn-Banach, assuming $\mathcal{X}$ is reflexive) $x \in \mathcal{X}$ such that $\hat{x}|_\mathcal{N} = 0$ (i.e. $x \in \mathcal{N}^\perp$) but $f(x) = \hat{x}(f) \neq 0$, so $f \notin (\mathcal{N}^\perp)^0$.

(c) Note that $\alpha = \pi^\dagger$, so $\alpha \in L((\mathcal{X}/\mathcal{M})^*, \mathcal{X}^*)$ and $\|\alpha\| = 1$ by Exercises 22(a) and 12(c). Since $\pi|_\mathcal{M} = 0$ it is clear that $\mathrm{Im}(\alpha) \subseteq \mathcal{M}^0$. Conversely, if $f \in \mathcal{M}^0$ then f factors through π by the universal property of quotient spaces (alternatively, check that $x + \mathcal{M} \mapsto f(x)$ is a well-defined element of $(\mathcal{X}/\mathcal{M})^*$), and hence $f \in \mathrm{Im}(\alpha)$. It remains to show that $\|\alpha(f)\| \geq \|f\|$ for all $f \in (\mathcal{X}/\mathcal{M})^*$. If $X \in \mathcal{X}/\mathcal{M}$ and $x \in X$ then

$$|f(X)| = |f(\pi(x))| \leq \|f \circ \pi\|\|x\|$$

and hence $|f(X)| \leq \|f \circ \pi\| \inf_{x \in X} \|x\| = \|\alpha(f)\|\|X\|$, so we are done.

(d) Clearly $\beta|_{\mathcal{M}^0} = 0$, so (by the universal property of quotient spaces) $\beta = \overline{\beta} \circ \pi$ for some $\overline{\beta} \in L(\mathcal{X}^*/\mathcal{M}^0, \mathcal{M}^*)$, where $\pi : \mathcal{X}^* \to \mathcal{X}^*/\mathcal{M}^0$ is the quotient map. If $F \in \mathcal{X}^*/\mathcal{M}^0$ and $f \in F$ then

$$\|\overline{\beta}(F)\| = \|\beta(f)\| = \sup_{x \in \mathcal{M}\setminus\{0\}} \frac{|f(x)|}{\|x\|} \leq \sup_{x \in \mathcal{X}\setminus\{0\}} \frac{|f(x)|}{\|x\|} = \|f\|$$

and hence $\|\overline{\beta}(F)\| \leq \inf_{f \in F} \|f\| = \|F\|$. Conversely, choose $f \in F$ and note that $x \mapsto \|\overline{\beta}(F)\|\|x\|$ is a seminorm on $\mathcal{X}$ such that

$$|f(x)| = |\beta(f)(x)| = |\overline{\beta}(F)(x)| \leq \|\overline{\beta}(F)\|\|x\|$$

for all $x \in \mathcal{M}$. By Hahn-Banach there exists $g \in \mathcal{X}^*$ such that $\|g\| \leq \|\overline{\beta}(F)\|$ and $g|_\mathcal{M} = f|_\mathcal{M}$. In particular $g \in F$, so $\|F\| \leq \|g\| \leq \|\overline{\beta}(F)\|$. Since β is surjective (Hahn-Banach), so is $\overline{\beta}$.

24. (a) Let $F \in (\mathcal{X}^*)\widehat{\ } \cap \widehat{\mathcal{X}}^0$. Then $F(\widehat{\mathcal{X}}) = \{0\}$ and there exists $f \in \mathcal{X}^*$ such that $F(g) = g(f)$ for all $g \in \mathcal{X}^{**}$. If $x \in \mathcal{X}$ then $f(x) = \widehat{x}(f) = F(\widehat{x}) = 0$, so $f = 0$ and hence $F(g) = g(0) = 0$ for all $g \in \mathcal{X}^{**}$. Therefore $F = 0$, which shows that $(\mathcal{X}^*)\widehat{\ } \cap \widehat{\mathcal{X}}^0 = \{0\}$.

Now let $F \in \mathcal{X}^{***}$. Define $f : \mathcal{X} \to K$ by $f(x) := F(\widehat{x})$. Note that $f \in \mathcal{X}^*$ because it is the composition of two bounded linear maps. Now $\widehat{f} \in (\mathcal{X}^*)\widehat{\ }$ and $\widehat{f}(\widehat{x}) = \widehat{x}(f) = f(x) = F(\widehat{x})$ for all $x \in \mathcal{X}$, so $F - \widehat{f} \in \widehat{\mathcal{X}}^0$ and hence $F = \widehat{f} + (F - \widehat{f}) \in (\mathcal{X}^*)\widehat{\ } + \widehat{\mathcal{X}}^0$. This shows that $(\mathcal{X}^*)\widehat{\ } + \widehat{\mathcal{X}}^0 = \mathcal{X}^{***}$.

(b) Suppose that $\mathcal{X}$ is reflexive. Then $\widehat{\mathcal{X}}^0 = \{F \in \mathcal{X}^{***} \mid F(\mathcal{X}^{**}) = \{0\}\} = \{0\}$, so $\mathcal{X}^{***} = (\mathcal{X}^*)\widehat{}$ by exercise (a). This shows that $\mathcal{X}^*$ is reflexive.

Conversely, if $\mathcal{X}^*$ is reflexive, then $\widehat{\mathcal{X}}^0 = \mathcal{X}^{***} \cap \widehat{\mathcal{X}}^0 = \{0\}$ by exercise (a). Since $\mathcal{X}$ is a Banach space, $\widehat{\mathcal{X}}$ is a closed subspace of $\mathcal{X}^{**}$. If there exists $f \in \mathcal{X}^{**} \setminus \widehat{\mathcal{X}}$ then there exists $g \in \mathcal{X}^{***}$ such that $g(f) \neq 0$ but $g(\widehat{\mathcal{X}}) = \{0\}$. This implies that g is a non-zero element of $\widehat{\mathcal{X}}^0$, which is a contradiction. Therefore $\mathcal{X}^{**} \setminus \widehat{\mathcal{X}} = \varnothing$, which shows that $\mathcal{X}$ is reflexive.

25. If $\mathcal{X} = \{0\}$ then it is clearly separable. Otherwise, let $\{f_n\}_{n=1}^\infty$ be a countable dense subset of $\mathcal{X}^*$. For each $n \in \mathbb{N}$ either $\|f_n\| = 0$ or $\|f_n\| > 0$. In the first case choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$, so that $|f_n(x_n)| = 0 = \frac{1}{2}\|f_n\|$. Otherwise $\frac{1}{2}\|f_n\| < \|f_n\| = \sup\{|f_n(x)| \mid x \in \mathcal{X}, \|x\| = 1\}$, so there exists $x_n \in \mathcal{X}$ such that $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Let $\mathcal{M}$ be the closed subspace of $\mathcal{X}$ generated by $\{x_n\}_{n=1}^\infty$, and suppose that there exists $x \in \mathcal{X} \setminus \mathcal{M}$. Then there exists $f \in \mathcal{X}^*$ such that $f(x) \neq 0$, $f(\mathcal{M}) = \{0\}$ and $\|f\| = 1$. Since $\{f_n\}_{n=1}^\infty$ is dense there exists $n \in \mathbb{N}$ such that $\|f - f_n\| < \frac{1}{3}$. It follows that

$$1 = \|f\| \leq \|f - f_n\| + \|f_n\| < \frac{1}{3} + 2|f_n(x_n)| = \frac{1}{3} + 2|f_n(x_n) - f(x_n)| \leq \frac{1}{3} + 2\|f_n - f\|\|x_n\| < \frac{1}{3} + \frac{2}{3} = 1,$$

which is a contradiction. Therefore $\mathcal{M} = \mathcal{X}$. Now let Q be a countable dense subset of K, and let $x \in \mathcal{X}$. Given $\varepsilon \in (0, \infty)$, there exist $n \in \mathbb{N} \cup \{0\}$ and $a_1, a_2, \ldots, a_n \in K$ such that $\sum_{k=1}^n a_k x_k \in B_{\varepsilon/2}(x)$. For each $k \in \{1, 2, \ldots, n\}$ choose $q_k \in Q$ with $|a_k - q_k| < \frac{\varepsilon}{2n}$. Then

$$\left\|\sum_{k=1}^n a_k x_k - \sum_{k=1}^n q_k x_k\right\| = \left\|\sum_{k=1}^n (a_k - q_k) x_k\right\| \leq \sum_{k=1}^n |a_k - q_k|\|x_k\| = \sum_{k=1}^n |a_k - q_k| \leq n\frac{\varepsilon}{2n} = \frac{\varepsilon}{2},$$

which implies that $\sum_{k=1}^n q_k x_k \in B_\varepsilon(x)$. Therefore the set of linear combinations of $\{x_n\}_{n=1}^\infty$ with coefficients in Q is dense in $\mathcal{X}$. This set is countable because it is an image of the countable set $\cup_{n=0}^\infty Q^n$. It follows that $\mathcal{X}$ is separable.

29. (a) Clearly $0 \in \mathcal{X}$. If $f, g \in \mathcal{X}$ and $a \in K$ then

$$\sum_{n=1}^\infty n|(f + ag)(n)| = \sum_{n=1}^\infty n|f(n) + ag(n)| \leq \sum_{n=1}^\infty n|f(n)| + \sum_{n=1}^\infty n|ag(n)| = \sum_{n=1}^\infty n|f(n)| + |a|\sum_{n=1}^\infty n|g(n)| < \infty,$$

so $\mathcal{X}$ is a subspace of $\mathcal{Y}$. Since $\sum_{n=1}^\infty n^{-2}$ converges but $\sum_{n=1}^\infty n^{-1}$ does not, the map $n \mapsto n^{-2}$ is in $\mathcal{Y} \setminus \mathcal{X}$. Let $f \in \mathcal{Y}$ and $\varepsilon \in (0, \infty)$. There exists $N \in \mathbb{N}$ such that $\sum_{n=N}^\infty |f(n)| < \varepsilon$. It follows that

$$\sum_{n=N}^\infty |f(n) - n^{-1}f(n)| = \sum_{n=N}^\infty (1 - n^{-1})|f(n)| \leq \sum_{n=N}^\infty |f(n)| < \varepsilon,$$

which implies that $\|f - g\| < \varepsilon$, where $g \in \mathcal{X}$ is defined by

$$g(n) := \begin{cases} f(n) & \text{if } n < N \\ n^{-1}f(n) & \text{if } n \geq N. \end{cases}$$

This shows that $\mathcal{X}$ is a proper dense subset of $\mathcal{Y}$, so $\overline{\mathcal{X}} = \mathcal{Y} \neq \mathcal{X}$ and hence $\mathcal{X}$ is not complete.

(b) Let $\langle f_n \rangle_{n=1}^\infty$ be a sequence in $\mathcal{X}$ which converges to $f \in \mathcal{X}$ such that $\langle Tf_n \rangle_{n=1}^\infty$ converges to $g \in \mathcal{Y}$. Given $\varepsilon \in (0, \infty)$, there exists $N \in \mathbb{N}$ such that $\sum_{n=N}^\infty n|f(n)| < \frac{\varepsilon}{4}$ and $\sum_{n=N}^\infty |g(n)| < \frac{\varepsilon}{4}$. Moreover, there exists $M \in \mathbb{N}$ such that $\|g - Tf_m\| < \frac{\varepsilon}{4}$ and $\|f - f_m\| < \frac{\varepsilon}{4}N^{-1}$ for all $m \in \mathbb{N}$ with $m \geq M$. It follows that

$$\sum_{n=1}^\infty |Tf(n) - Tf_m(n)| = \sum_{n=1}^{N-1} |nf(n) - nf_m(n)| + \sum_{n=N}^\infty |nf(n) - Tf_m(n)|$$

$$\begin{aligned}
&\leq \sum_{n=1}^{N-1} n|f(n) - f_m(n)| + \sum_{n=N}^{\infty} n|f(n)| + \sum_{n=N}^{\infty} |Tf_m(n)| \\
&< \sum_{n=1}^{N-1} N|f(n) - f_m(n)| + \frac{\varepsilon}{4} + \sum_{n=N}^{\infty} |Tf_m(n) - g(n)| + \sum_{n=N}^{\infty} |g(n)| \\
&< N\|f - f_m\| + \frac{\varepsilon}{4} + \|Tf_m - g\| + \frac{\varepsilon}{4} \\
&< \varepsilon
\end{aligned}$$

for all $m \in \mathbb{N}$ with $m \geq M$. Therefore $\langle Tf_n\rangle_{n=1}^{\infty}$ converges to Tf, so T is closed.

For each $m \in \mathbb{N}$ define $f_m \in \mathcal{X}$ by

$$f_m(n) := \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

Then $\|f_m\| = 1$ but

$$\|Tf_m\| = \sum_{n=1}^{\infty} |Tf_m(n)| = \sum_{n=1}^{\infty} |nf_m(n)| = \sum_{n=1}^{\infty} nf_m(n) = m$$

for all $m \in \mathbb{N}$. This shows that T is not bounded.

(c) Clearly $Sf(n) = n^{-1}f(n)$ for all $f \in \mathcal{Y}$ and $n \in \mathbb{N}$. It follows that

$$\|Sf\| = \sum_{n=1}^{\infty} |Sf(n)| = \sum_{n=1}^{\infty} |n^{-1}f(n)| = \sum_{n=1}^{\infty} n^{-1}|f(n)| \leq \sum_{n=1}^{\infty} |f(n)| = \|f\|$$

for all $f \in \mathcal{Y}$, so S is bounded. Since $S = T^{-1}$, it is obvious that S is surjective. If S were open then T would be continuous, contradicting part (b).

32. The identity map from $(\mathcal{X}, \|\cdot\|_2)$ to $(\mathcal{X}, \|\cdot\|_1)$ is a bijective bounded linear map between Banach spaces, so it is a homeomorphism by the open mapping theorem. It follows that the identity map from $(\mathcal{X}, \|\cdot\|_1)$ to $(\mathcal{X}, \|\cdot\|_2)$ is bounded, so there exists $M \in [0, \infty)$ such that $\|\cdot\|_2 \leq M\|\cdot\|_1$. This shows that the two norms are equivalent.

33. Suppose there is such a sequence $(a_n)_{n=1}^{\infty}$, and let $T : B(\mathbb{N}) \to L^1(\mu)$ be the corresponding map. For each $n \in \mathbb{N}$ multiplication by a_n is a linear map $\mathbb{C} \to \mathbb{C}$, and the product of these maps gives a linear map $T : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$. As each a_n is positive, it is clear that T is injective. If $(c_n)_{n=1}^{\infty}$ is bounded then $(a_nc_n)_{n=1}^{\infty}$ is in $L^1(\mu)$. Conversely, if $(c_n)_{n=1}^{\infty}$ is in $L^1(\mu)$, then $\sum_{n=1}^{\infty} a_n|\frac{c_n}{a_n}| < \infty$, so $(\frac{c_n}{a_n})_{n=1}^{\infty}$ is bounded. This shows that T restricts to a bijection $B(\mathbb{N}) \to L^1(\mu)$. The restriction is bounded because $\sum_{n=1}^{\infty} |a_nc_n| \leq \sum_{n=1}^{\infty} a_n\|c\|_{\infty} = \|a\|_1\|c\|_{\infty}$ for every bounded sequence $(c_n)_{n=1}^{\infty}$ (this is a special case of Hölder's inequality; we know that $\|a\|_1 < \infty$ because $(1)_{n=1}^{\infty}$ is bounded). Hence, by the open mapping theorem, $T^{-1} : L^1(\mu) \to B(\mathbb{N})$ is bounded. For each $k \in \mathbb{N}$

$$\frac{1}{a_k} = \left\| \left(\frac{\delta_{kn}}{a_n}\right)_{n=1}^{\infty} \right\|_u \leq \|T^{-1}\| \sum_{n=1}^{\infty} |\delta_{kn}| = \|T^{-1}\|,$$

which implies that $\sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{\infty} \|T^{-1}\| = \infty$. This is impossible, so no such sequence exists.

34. (a) Let $\langle f_n\rangle_{n=1}^{\infty}$ be a sequence in $L_k^1([0,1])$ which converges in $L_k^1([0,1])$ to $f \in L_k^1([0,1])$ and also converges in $C^{k-1}([0,1])$ to $g \in C^{k-1}([0,1])$. Suppose that $f \neq g$. There exists $x \in [0,1]$ such that $f(x) \neq g(x)$, and hence

there exists $\varepsilon \in (0, \infty)$ such that $|f(x) - g(x)| > 2\varepsilon$. Since f and g are continuous, there exists $\delta \in (0, \infty)$ such that $|f(y) - g(y)| > \varepsilon$ for all $y \in [0, 1] \cap (-\delta, \delta)$. Clearly $m([0, 1] \cap (-\delta, \delta)) \geq \delta$, so

$$\int_0^1 |f - g| \geq \delta\varepsilon.$$

However, there exists $N \in \mathbb{N}$ such that $\|f - f_N\|_{L^1_k} < \frac{\delta\varepsilon}{2}$ and $\|g - f_N\|_{C^{k-1}} < \frac{\delta\varepsilon}{2}$, and hence

$$\delta\varepsilon \leq \int_0^1 |f - g| \leq \int_0^1 |f - f_N| + \int_0^1 |f_N - g| \leq \|f - f_N\|_{L^1_k} + \|g - f_N\|_u < \frac{\delta\varepsilon}{2} + \|g - f_N\|_{C^{k-1}} < \delta\varepsilon.$$

This is impossible, so in fact $f = g$. This shows that the inclusion $L^1_k([0, 1]) \hookrightarrow C^{k-1}([0, 1])$ is closed, so it is continuous by the closed graph theorem.

(b) Let $f \in L^1_k([0, 1])$ and suppose that $\|f\|_{L^1_k} = 1$. Also let $j \in \{0, \ldots, k - 1\}$. Then $f^{(j)}$ is absolutely continuous, so for all $x \in [0, 1]$

$$|f^{(j)}(x) - f^{(j)}(0)| = \left|\int_0^x f^{(j+1)}\right| \leq \int_0^x |f^{(j+1)}| \leq \int_0^1 |f^{(j+1)}| \leq \|f\|_{L^1_k} = 1.$$

Moreover,

$$|f^{(j)}(0)| = \int_0^1 |f^{(j)}(0)|\, dx \leq \int_0^1 |f^{(j)}(0) - f^{(j)}(x)|\, dx + \int_0^1 |f^{(j)}(x)|\, dx \leq \int_0^1 1\, dx + \|f\|_{L^1_k} = 2$$

and hence

$$|f^{(j)}(x)| \leq |f^{(j)}(x) - f^{(j)}(0)| + |f^{(j)}(0)| \leq 1 + 2 = 3$$

for all $x \in [0, 1]$. This implies that $\|f^{(j)}\|_u \leq 3$, in which case $\|f\|_{C^{k-1}} \leq 3(k - 1)$. This shows that the inclusion $L^1_k([0, 1]) \hookrightarrow C^{k-1}([0, 1])$ is bounded as a linear map, so it is continuous.

37. Define $T' : \mathcal{Y}^* \to \mathcal{X}^*$ by $T'f := f \circ T$. Note that T' is well-defined because $f \circ T \in \mathcal{X}^*$ for all $f \in \mathcal{Y}^*$. Moreover, T' is linear for the same reason that the adjoint of a bounded linear map is linear. Set $\mathcal{A} := \{\widehat{x} \circ T' \mid x \in \mathcal{X}, \|x\| = 1\}$. If $x \in \mathcal{X}$, $f \in \mathcal{Y}^*$ and $\|x\| = \|f\| = 1$ then

$$|\widehat{x}(T'f)| = |T'f(x)| = |f(Tx)| \leq \|f\|\|T(x)\| = \|T(x)\|,$$

which implies that $\widehat{x} \circ T'$ is bounded. Therefore $\mathcal{A}$ is a subset of $L(\mathcal{Y}^*, K)$. Since

$$\sup_{S \in \mathcal{A}} |Sf| = \sup_{x \in \mathcal{X},\, \|x\|=1} |\widehat{x}(T'f)| = \sup_{x \in \mathcal{X},\, \|x\|=1} |T'f(x)| = \|T'f\| < \infty$$

for all $f \in \mathcal{Y}^*$, the uniform boundedness principle implies that $M := \sup_{S \in \mathcal{A}} \|S\| < \infty$. Let $x \in \mathcal{X}$ with $\|x\| = 1$ and $Tx \neq 0$. By Hahn-Banach there exists $g \in \mathcal{Y}^*$ such that $\|g\| = 1$ and $g(Tx) = \|Tx\|$. It follows that

$$\|Tx\| = g(Tx) = |g(Tx)| \leq \sup_{f \in \mathcal{Y}^*,\, \|f\|=1} |f(Tx)| = \sup_{f \in \mathcal{Y}^*,\, \|f\|=1} |T'f(x)| = \sup_{f \in \mathcal{Y}^*,\, \|f\|=1} |\widehat{x}(T'f)| = \|\widehat{x} \circ T'\| \leq M.$$

This also holds if we allow $Tx = 0$, which shows that T is bounded.

38. Since addition and scalar multiplication respect limits, T is linear. Every convergent sequence in a metric space is bounded, so $\sup_{n \in \mathbb{N}} \|T_n x\| < \infty$ for each $x \in \mathcal{X}$. Therefore $M := \sup_{n \in \mathbb{N}} \|T_n\| < \infty$ by the uniform boundedness principle. It follows that T is bounded, since for each $x \in \mathcal{X}$

$$\|Tx\| = \left\|\lim_{n \to \infty} T_n x\right\| = \lim_{n \to \infty} \|T_n x\| \leq \sup_{n \in \mathbb{N}} \|T_n x\| \leq \sup_{n \in \mathbb{N}} \|T_n\|\|x\| = M\|x\|.$$

40. If $k \in \mathbb{N}$, then $\sup_{j\in\mathbb{N}} \|T_{jk}\| = \infty$ (otherwise $\|T_{jk}x\| \leq \sup_{j\in\mathbb{N}} \|T_{jk}\|\|x\| < \infty$ for all $j \in \mathbb{N}$ and $x \in \mathcal{X}$), and hence (by the uniform boundedness principle) the set $E_k := \{x \in \mathcal{X} \mid \sup_{j\in\mathbb{N}} \|T_{jk}x\| < \infty\}$ is meager. It follows that $\cup_{k=1}^{\infty} E_k$ is meager, so $\cup_{k=1}^{\infty} E_k \neq \mathcal{X}$. Hence, there exists $x \in \mathcal{X}$ such that $\sup_{j\in\mathbb{N}} \|T_{jk}x\| = \infty$ for all $k \in \mathbb{N}$.

41. Fix a norm on $\mathcal{X}$. Let $\{x_n\}_{n=1}^{\infty}$ be a (Hamel) basis for $\mathcal{X}$, so that $\mathcal{X} = \cup_{N=1}^{\infty} \operatorname{span}\{x_n\}_{n=1}^{N}$. Given $N \in \mathbb{N}$, the subspace $\operatorname{span}\{x_n\}_{n=1}^{N}$ is closed by exercise 18(b). If $\mathcal{Y}$ is a subspace of $\mathcal{X}$ with non-empty interior, there exists $r \in (0, \infty)$ and $y \in \mathcal{Y}$ such that $B_r(y) \subseteq \mathcal{Y}$. Since $\mathcal{Y}$ is closed under subtraction, it follows that $B_r(0) \subseteq \mathcal{Y}$. This implies that $\mathcal{Y} = \mathcal{X}$, because $\mathcal{Y}$ is closed under scalar multiplication. Therefore, the proper closed subspace $\operatorname{span}\{x_n\}_{n=1}^{N}$ of $\mathcal{X}$ is nowhere dense. This shows that $\mathcal{X}$ is the countable union of nowhere dense sets, so $\mathcal{X}$ is not complete by the Baire category theorem.

43. (a) Suppose there is a nonzero $x \in \mathcal{X}$ such that $p_\alpha(x) = 0$ for all $\alpha \in A$. If $\alpha \in A$ and $\varepsilon \in (0, \infty)$ then $0 \in U_{x\alpha\varepsilon}$, so 0 lies in every finite intersection of sets of this form. By Theorem 5.14(a), it follows that 0 lies in every neighbourhood of x, so $\mathcal{X}$ is not Hausdorff (or even T_0, by a similar argument).

    Conversely, suppose that, for each nonzero $x \in \mathcal{X}$, there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$. If $x, y \in \mathcal{X}$ are distinct, then $x - y \neq 0$ and hence $p_\alpha(x - y) \neq 0$ for some $\alpha \in A$. Set $\varepsilon := \frac{p_\alpha(x-y)}{2}$, and note that $U_{x\alpha\varepsilon} \cap U_{y\alpha\varepsilon} = \varnothing$; indeed $p_\alpha(x - y) \leq p_\alpha(x - z) + p_\alpha(z - y)$ for all $z \in \mathcal{X}$, so $p_\alpha(x - z)$ and $p_\alpha(z - y)$ cannot both be less than ε. This implies that $\mathcal{X}$ is Hausdorff.

    (b) Replace A by $\mathbb{N}$ (if A is finite, extend it by adding copies of the zero seminorm) and define $\rho : \mathcal{X}^2 \to [0, \infty)$ by $\rho(x, y) := \sum_{n=1}^{\infty} 2^{-n}\Phi(p_n(x - y))$, where Φ is the function from Exercise 4.56. If $x, y \in \mathcal{X}$ and $\rho(x, y) = 0$ then $\Phi(p_n(x - y)) = 0$ and hence $p_n(x - y) = 0$ for all $n \in \mathbb{N}$, so $x - y = 0$ by part (a). It is clear that ρ has all the other properties of a translation-invariant metric. If $x \in \mathcal{X}$, $n \in \mathbb{N}$, $\varepsilon \in (0, \infty)$ and $y \in U_{xn\varepsilon}$, then $\delta := 2^{-n}\Phi(\varepsilon - p_n(y - x))$ is positive and $B_\delta(y) \subseteq U_{xn\varepsilon}$ by the triangle inequality. This shows that $U_{xn\varepsilon}$ is open with respect to ρ. Conversely, let $x \in \mathcal{X}$, $\varepsilon \in (0, \infty)$ and $y \in B_\varepsilon(x)$. If $\varepsilon \geq 1$ then $B_\varepsilon(x) = \mathcal{X}$, which is open in $\mathcal{X}$. Otherwise choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \varepsilon - \rho(x, y)$, set $\delta := \Phi^{-1}(\frac{\varepsilon-\rho(x,y)}{2})$ and note that $\cap_{n=1}^{N} U_{yn\delta} \subseteq B_\varepsilon(x)$. This shows that $B_\varepsilon(x)$ is open in $\mathcal{X}$, so $\mathcal{X}$ is metrisable (with respect to ρ).

44. Let $\langle x_\alpha\rangle_{\alpha\in A}$ be a Cauchy net in $\mathcal{X}$, and $\{U_n\}_{n=1}^{\infty}$ a nested neighbourhood base of $0 \in \mathcal{X}$. For each $n \in \mathbb{N}$ there exist $\alpha_n, \beta_n \in A$ such that $x_\alpha - x_\beta \in U_n$ for all $\alpha, \beta \in A$ with $\alpha \gtrsim \alpha_n$ and $\beta \gtrsim \beta_n$. Without loss of generality assume that $\alpha_{n-1} \lesssim \alpha_n = \beta_n$ (take an upper bound of $\{\alpha_{n-1}, \alpha_n, \beta_n\}$, and ignore the condition $\alpha_0 \lesssim \alpha_1$). If $U \subseteq \mathcal{X}$ is a neighbourhood of 0, then $U_N \subseteq U$ for some $N \in \mathbb{N}$, and $x_{\alpha_m} - x_{\alpha_n} \in U_N$ for all $m, n \in \mathbb{N}$ with $m, n \geq N$. This shows that $(x_{\alpha_n})_{n=1}^{\infty}$ is a Cauchy sequence, so it has a limit $x \in \mathcal{X}$. If $U \subseteq \mathcal{X}$ is a neighbourhood of x, its preimage in $\mathcal{X}^2$ under addition is a neighbourhood of $(0, x)$. In particular, there exists $N \in \mathbb{N}$ and a neighbourhood V of x such that $U_N \times V$ maps into U under addition. Without loss of generality $x_{\alpha_n} \in V$ for all $n \in \mathbb{N}$ with $n \geq N$. If $\alpha \in A$ and $\alpha \gtrsim \alpha_N$, then $x_\alpha - x_{\alpha_N} \in U_N$ and $x_{\alpha_N} \in V$, so $x_\alpha = x_\alpha - x_{\alpha_N} + x_{\alpha_N} \in U$. This shows that $\langle x_\alpha\rangle_{\alpha\in A}$ converges to x.

45. For each $n, k \in \mathbb{N} \cup \{0\}$ set $K_n := [-n, n]$ and define a seminorm $p_{n,k}$ on $C^\infty(\mathbb{R})$ by $p_{n,k}(f) := \|f^{(k)}|_{K_n}\|_u$ (this is indeed a seminorm because differentiation and restriction are linear maps). If $f \in C^\infty(\mathbb{R})$ is nonzero then $f|_{K_n} \neq 0$ for sufficiently large $n \in \mathbb{N}$, so $p_{n,0}(f) \neq 0$. By Exercise 43(a) $C^\infty(\mathbb{R})$, with the topology defined by these seminorms, is Hausdorff. Every sequence in $C^\infty(\mathbb{R})$ whose derivatives of every order converge uniformly on compact sets to the derivatives of some $f \in C^\infty(\mathbb{R})$ also converges (to f) in this topology, by Theorem 5.14(b). The converse holds because every compact subset of $\mathbb{R}$ is contained in some K_n. By Exercise 43(b) $C^\infty(\mathbb{R})$ is metrisable, hence first countable. It therefore remains to show that every Cauchy sequence in $C^\infty(\mathbb{R})$ converges. If $(f_j)_{j=1}^{\infty}$ is a Cauchy sequence, then

$(f_j^{(k)}|_{K_n})_{n=1}^{\infty}$ is uniformly Cauchy by Theorem 5.14(b), and hence uniformly convergent, for each $n, k \in \mathbb{N} \cup \{0\}$. By taking pointwise limits it is clear that the limits of these sequences agree, namely for each $k \in \mathbb{N} \cup \{0\}$ there exists $g_k \in C(\mathbb{R})$ such that $(f_j^{(k)})_{n=1}^{\infty}$ converges to g_k uniformly on compact sets. If $x \in \mathbb{R}$ then, by the dominated convergence theorem,

$$g_0(x) = \lim_{j\to\infty} f_j(x) = \lim_{j\to\infty} \int_0^x f_j' = \int_0^x g_1.$$

By the fundamental theorem of calculus, it follows that $g_0' = g_1$; similarly $g_0^{(k)} = g_k$ for all $k \in \mathbb{N}$. Therefore $(f_j)_{n=1}^{\infty}$ converges to $g_0 \in C^{\infty}(\mathbb{R})$, and $C^{\infty}(\mathbb{R})$ is complete.

47. (a) If $\langle T_n\rangle_{n=1}^{\infty}$ converges to T strongly, then it converges to T weakly. Hence, it suffices to consider only the weak case. Fix $x \in \mathcal{X}$, and let $f \in \mathcal{Y}^*$. Then $\langle f(T_n x)\rangle_{n=1}^{\infty}$ converges (to $f(Tx)$), and hence $\sup_{n\in\mathbb{N}} |f(T_n x)| < \infty$. In other words $\sup_{n\in\mathbb{N}} |\widehat{x}(T_n^{\dagger} f)| = \sup_{n\in\mathbb{N}} |\widehat{x}(f \circ T_n)| < \infty$. By the uniform boundedness principle, it follows that $M := \sup_{n\in\mathbb{N}} \|\widehat{x} \circ T_n^{\dagger}\| < \infty$. Given $n \in \mathbb{N}$, either $T_n x = 0$ or (by Hahn-Banach) there exists $g \in \mathcal{Y}^*$ such that $\|g\| = 1$ and $g(T_n x) = |g(T_n x)| = \|T_n x\|$. In the first case $\|T_n x\| = 0 \leq M$, and otherwise

$$\|T_n x\| = |g(T_n x)| \leq \sup_{f\in\mathcal{Y}^*,\, \|f\|=1} |f(T_n x)| = \sup_{f\in\mathcal{Y}^*,\, \|f\|=1} |T_n^{\dagger} f(x)| = \sup_{f\in\mathcal{Y}^*,\, \|f\|=1} |\widehat{x}(T_n^{\dagger} f)| = \|\widehat{x} \circ T_n^{\dagger}\| \leq M.$$

This shows that $\sup_{n\in\mathbb{N}} \|T_n x\| \leq M < \infty$, so $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$ by the uniform boundedness principle.

(b) Let $\langle x_n\rangle_{n=1}^{\infty}$ by a weakly convergent sequence in $\mathcal{X}$, and let $f \in \mathcal{X}^*$. Then $\langle f(x_n)\rangle_{n=1}^{\infty}$ converges in K, and hence $\sup_{n\in\mathbb{N}} |f(x_n)| < \infty$. In other words $\sup_{n\in\mathbb{N}} |\widehat{x}_n(f)| < \infty$, in which case $\sup_{n\in\mathbb{N}} \|\widehat{x}_n\| < \infty$ by the uniform boundedness principle. Since $\widehat{\cdot}$ is an isometry, this shows that $\langle x_n\rangle_{n=1}^{\infty}$ is bounded.

The weak* topology on $\mathcal{X}^*$ is the same as the strong operator topology, so every weak*-convergent sequence in $\mathcal{X}^*$ is bounded by part (a).

48. (a) Let $\langle x_\alpha\rangle_{\alpha\in A}$ be a net in B which converges weakly to $x \in \mathcal{X}$. If $f \in \mathcal{X}^*$ and $\|f\| = 1$ then

$$|\widehat{x}(f)| = |f(x)| = \left|\lim_{\alpha\in A} f(x_\alpha)\right| = \lim_{\alpha\in A} |f(x_\alpha)| \in [0, 1]$$

because $|\cdot|$ is continuous and $0 \leq |f(x_\alpha)| = |\widehat{x}_\alpha(f)| \leq \|\widehat{x}_\alpha\|\|f\| = \|x_\alpha\| \leq 1$ for all $\alpha \in A$. This shows that $\|x\| = \|\widehat{x}\| \leq 1$, so $x \in B$ and hence B is weakly closed.

(b) Choose $r \in (0, \infty)$ such that $E \subseteq B_r[0]$. Then $r^{-1}E \subseteq B$, so $\overline{r^{-1}E} \subseteq B$ where $\overline{r^{-1}E}$ is the weak closure of $r^{-1}E$. Since the weak topology makes $\mathcal{X}$ a toplogical vector space, multiplication by r^{-1} is weakly continuous and hence $r^{-1}\overline{E} \subseteq \overline{r^{-1}E} \subseteq B$. This implies that $\overline{E} \subseteq B_r[0]$, which shows that $\overline{E}$ is bounded.

(c) Choose $r \in (0, \infty)$ such that $F \subseteq B_r[0]$. Let f be a functional in the weak* closure of F. There exists a net $\langle f_\alpha\rangle_{\alpha\in A}$ in F which weak*-converges to f. If $x \in \mathcal{X}$ and $\|x\| = 1$ then

$$|f(x)| = \left|\lim_{\alpha\in A} f_\alpha(x)\right| = \lim_{\alpha\in A} |f_\alpha(x)| \in [0, r]$$

because $|\cdot|$ is continuous and $0 \leq |f_\alpha(x)| \leq \|f_\alpha\|\|x\| = \|f_\alpha\| \leq r$ for all $\alpha \in A$. This implies that

$$\|f\| = \sup_{x\in X,\, \|x\|=1} |f(x)| \leq r,$$

which shows that the weak* closure of F is bounded.

(d) Let $\langle f_n\rangle_{n=1}^\infty$ be a weak*-Cauchy sequence in $\mathcal{X}^*$. Then $\langle f_m - f_n\rangle_{(m,n)\in\mathbb{N}^2}$ weak*-converges to 0. Given $x \in \mathcal{X}$, it follows that $\langle f_m(x) - f_n(x)\rangle_{(m,n)\in\mathbb{N}^2}$ converges to 0, which means that for every $\varepsilon \in (0,\infty)$ there exists $(M,N) \in \mathbb{N}^2$ such that $|f_m(x) - f_n(x)| < \varepsilon$ for all $m,n \in \mathbb{N}$ with $m \geq M$ and $n \geq N$. This implies that $\langle f_n(x)\rangle_{n=1}^\infty$ is a Cauchy sequence in the usual sense (by taking $\max\{M,N\}$), so it has a limit $f(x) \in K$. By exercise 38, it follows that $f \in \mathcal{X}^*$. Since $\langle f_n\rangle_{n=1}^\infty$ converges to f pointwise, it weak*-converges to f.

49. (a) Let $E \subseteq \mathcal{X}$ be non-empty and weakly open. Choose $x \in E$. Then there exist $\varepsilon_1,\ldots,\varepsilon_n \in (0,\infty)$ and $f_1,\ldots,f_n \in \mathcal{X}^*$ such that $\cap_{k=1}^n\{y \in \mathcal{X} \mid |f_k(x)-f_k(y)| < \varepsilon_k\} \subseteq E$. Since $\mathcal{X}$ is infinite-dimensional, we may choose $x_1,\ldots,x_{2^n} \in \mathcal{X}$ so that $\{x_k\}_{k=1}^{2^n}$ is linearly independent. If $k \in \{1,\ldots,2^{n-1}\}$ then

$$f_1(x_{2k})x_{2k-1} - f_1(x_{2k-1})x_{2k} \in \ker(f_1)$$

is zero only if $f_1(x_{2k}) = f_1(x_{2k-1}) = 0$. In either case there exists $x'_k \in \operatorname{span}\{x_{2k-1},x_{2k}\}$ such that $x'_k \neq 0$ and $f_1(x'_k) = 0$. Thus $\{x'_1,\ldots,x'_{2^{n-1}}\}$ is a linearly independent subset of $\ker(f_1)$. Applying this construction again gives non-zero vectors $x''_1,\ldots,x''_{2^{n-2}} \in \ker(f_2)$ such that $x''_k \in \operatorname{span}\{x'_{2k-1},x'_{2k}\} \subseteq \ker(f_1)$ for all $k \in \{1,\ldots,2^{n-2}\}$. Continuing inductively, we eventually obtain $x_1^{(n)} \in \cap_{k=1}^n \ker(f_k)$ such that $x_1^{(n)} \neq 0$. Set $y := \frac{x_1^{(n)}}{\|x_1^{(n)}\|}$. If $n \in \mathbb{N}$ then $f_k(x-ny) = f_k(x) - nf_k(y) = f_k(x)$ for all $k \in \{1,\ldots,n\}$, which implies that

$$x - ny \in \cap_{k=1}^n\{y \in \mathcal{X} \mid |f_k(x) - f_k(y)| < \varepsilon_k\} \subseteq E.$$

Moreover $n = \|ny\| \leq \|ny - x\| + \|x\|$ and hence $\|x - ny\| \geq n - \|x\|$, which implies that E is unbounded.

Now let $F \subseteq \mathcal{X}^*$ be non-empty and weak*-open. Every sequence in F^c which converges weakly to $f \in \mathcal{X}^*$ also weak*-converges to f, so $f \in F^c$ and hence F^c is weakly closed. Therefore F is weakly open, so it is unbounded by the previous argument (note that $\mathcal{X}^*$ is infinite-dimensional, because otherwise $\mathcal{X}^{**}$ would be a finite-dimensional space with an infinite-dimensional subspace $\widehat{\mathcal{X}}$).

(b) Let $E \subseteq \mathcal{X}$ be bounded. The weak closure of E is bounded by exercise 48(b), so it has empty interior by part (a). Similarly, if $F \subseteq \mathcal{X}^*$ is bounded, its weak* closure is bounded by exercise 48(c), and thus has empty interior by part (a). Therefore E and F are nowhere dense in the weak and weak* topologies.

(c) This follows immediately from part (b), because $\mathcal{X} = \cup_{n\in\mathbb{N}}B_n[0]$ and $\mathcal{X}^* = \cup_{n\in\mathbb{N}}B_n[0]$.

(d) Suppose the weak* topology on $\mathcal{X}^*$ was defined by some translation-invariant metric ρ. Let $\langle f_n\rangle_{n=1}^\infty$ be a Cauchy sequence in the metric space $(\mathcal{X}^*,\rho)$. Given a weak*-neighbourhood U of 0, there exists $\varepsilon \in (0,\infty)$ such that $B_\varepsilon^\rho(0) \subseteq U$. Moreover, there exists $N \in \mathbb{N}$ such that $\rho(f_m,f_n) < \varepsilon$ for all $m,n \in \mathbb{N}$ with $m \geq N$ and $n \geq N$. This implies that $\rho(f_m - f_n, 0) < \varepsilon$, and hence $f_m - f_n \in B_\varepsilon^\rho(0) \subseteq U$, for all $(m,n) \in \mathbb{N}^2$ with $m \geq N$ and $n \geq N$. This means that $\langle f_m - f_n\rangle_{(m,n)\in\mathbb{N}^2}$ weak*-converges to 0, so $\langle f_n\rangle_{n=1}^\infty$ is a weak*-Cauchy sequence in $\mathcal{X}^*$, which weak*-converges by exercise 48(d). Therefore $\langle f_n\rangle_{n=1}^\infty$ converges in $(\mathcal{X}^*,\rho)$, which shows that $(\mathcal{X}^*,\rho)$ is a complete metric space. This contradicts the Baire category theorem, by part (c). Therefore, the weak* topology on $\mathcal{X}^*$ is not defined by a translation-invariant metric.

50. Let B be the closed unit ball in $\mathcal{X}^*$, and choose dense subsets $D \subseteq \mathcal{X}$ and $E \subseteq K$. The collection of finite intersections of sets of the form $\widehat{x}^{-1}(B_q(c)) \cap B$, where $x \in D$, $q \in \mathbb{Q} \cap (0,\infty)$ and $c \in E$, is clearly countable. If $U \subseteq B$ is open and $f \in U$ then $f \in \cap_{k=1}^n \widehat{x}_k^{-1}(U_k) \cap B \subseteq U$ for some $x_1,\ldots,x_n \in \mathcal{X}$ and open sets $U_1,\ldots,U_n \subseteq K$ (by definition of the weak* topology). It therefore suffices to show the following: given $x \in \mathcal{X}$ and $U \subseteq K$ open such that $f(x) \in U$, there exist $y \in D$, $q \in \mathbb{Q} \cap (0,\infty)$ and $c \in E$ such that $f \in \widehat{y}^{-1}(B_q(c)) \cap B \subseteq \widehat{x}^{-1}(U) \cap B$.

To this end, choose $r \in (0, \infty)$ such that $B_{3r}(f(x)) \subseteq U$, $y \in B_r(x) \cap D$, $q \in \mathbb{Q} \cap (0, r)$ and $c \in B_q(f(x)) \cap E$. Clearly $f \in \widehat{y}^{-1}(B_q(c))$. Moreover, if $g \in \widehat{y}^{-1}(B_q(c))$ and $\|g\| \leq 1$ then

$$|g(x) - f(x)| \leq |g(x) - g(y)| + |g(y) - c| + |c - f(x)| < \|g\|\|x - y\| + q + r < 3r$$

and hence $g \in \widehat{x}^{-1}(U)$. This shows that $f \in \widehat{y}^{-1}(B_q(c)) \cap B \subseteq \widehat{x}^{-1}(U) \cap B$, so B is second countable. It is also normal, because it is compact and Hausdorff, and hence metrisable.

51. Every weakly closed subset of $\mathcal{X}$ is norm-closed, because the weak topology is weaker than the norm topology (every element of $\mathcal{X}^*$ is norm-continuous, by definition). Conversely, let $\mathcal{M} \subseteq \mathcal{X}$ be a norm-closed subspace. If $x \in \mathcal{X} \setminus \mathcal{M}$, there exists (by Hahn-Banach) $f_x \in \mathcal{X}^*$ such that $f_x|_{\mathcal{M}} = 0$ and $f_x(x) \neq 0$. It is clear that $\mathcal{M} = \cap_{x \in \mathcal{X} \setminus \mathcal{M}} \ker(f_x)$, which is weakly closed because each f_x is weakly continuous.

52. (a) Obviously $\mathcal{M} \subseteq \mathcal{N}^0$. If $f \in \mathcal{N}^0$, then the image of $(T, f) : \mathcal{X} \to \mathbb{C}^{n+1}$ is a (closed) subspace not containing e_{n+1}. By Hahn-Banach there exists $\lambda \in (\mathbb{C}^{n+1})^*$ such that $\lambda|_{\mathrm{Im}(T,f)} = 0$ but $\lambda(e_{n+1}) \neq 0$. If $x \in \mathcal{X}$ then

$$0 = \lambda(f_1(x), \ldots, f_n(x), f(x)) = \sum_{i=1}^{n} f_i(x)\lambda(e_i) + f(x)\lambda(e_{n+1}),$$

and hence $f = -\sum_{i=1}^{n} \frac{\lambda(e_i)}{\lambda(e_{n+1})} f_i \in \mathcal{M}$. It follows from Exercise 23 that $\mathcal{M} \cong (\mathcal{X}/\mathcal{N})^*$.

(b) Let $\varepsilon \in (0, \infty)$ and $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{N}$ the quotient map. By Exercise 23 $\pi^\dagger : (\mathcal{X}/\mathcal{N})^* \to \mathcal{M}$ is an isometric isomorphism, so $\pi^{\dagger\dagger} : \mathcal{M}^* \to (\mathcal{X}/\mathcal{N})^{**}$ is also an isomorphism ($\dagger$ is a functor). Note that $\pi^{\dagger\dagger}(F|_{\mathcal{M}}) = \widehat{X}$ for some $X \in \mathcal{X}/\mathcal{N}$ (the embedding $\mathcal{X}/\mathcal{N} \to (\mathcal{X}/\mathcal{N})^{**}$ is between spaces of the same dimension). For each $i \in \{1, \ldots, n\}$ there exists $\overline{f}_i \in (\mathcal{X}/\mathcal{N})^*$ such that $\overline{f}_i \circ \pi = f_i$ (by the universal property of quotient spaces), so $F(f_i) = F(\pi^\dagger(\overline{f}_i)) = \pi^{\dagger\dagger}(F|_{\mathcal{M}})(\overline{f}_i) = \overline{f}_i(X) = f_i(x)$ for all $x \in X$. It remains to show that $\|x\| \leq (1 + \varepsilon)\|F\|$ for some $x \in X$; this follows from the fact that $\|X\| = \|\widehat{X}\| = \|F \circ \pi^\dagger\| \leq \|F\|\|\pi^\dagger\| = \|F\|$.

(c) If $f \in \mathcal{X}^*$ and $x \in \mathcal{X}$ then, by definition $f(\widehat{x}) = f(x) = \widehat{x}(f)$, so we identify f with $\widehat{f}|_{\widehat{\mathcal{X}}}$, which is continuous in the relative weak* topology on $\widehat{\mathcal{X}}$, because $\widehat{f}$ is continuous in the weak* topology on $\mathcal{X}^{**}$. Thus, the weak topology on $\widehat{\mathcal{X}}$ is weaker than the relative weak* topology.

Conversely, note that the weak* topology on $\mathcal{X}^{**}$ is generated by sets of the form $\widehat{f}^{-1}(U)$, where $f \in \mathcal{X}^*$ and $U \subseteq K$ is open. It follows from Proposition 4.4 that the relative weak* topology on $\widehat{\mathcal{X}}$ is generated by sets of the form $\widehat{f}^{-1}(U) \cap \widehat{\mathcal{X}}$. In particular, it suffices to show that $\widehat{f}|_{\widehat{\mathcal{X}}}$ is continuous in the weak topology, for all $f \in \mathcal{X}^*$. If $x \in \mathcal{X}$ then, by definition $\widehat{f}(x) = \widehat{f}(\widehat{x}) = \widehat{x}(f) = f(x)$, so we identify $\widehat{f}|_{\widehat{X}}$ with the continuous functional f.

(d) Let $F \in \mathcal{X}^{**}$, $\varepsilon_1, \ldots, \varepsilon_n \in (0, \infty)$ and $f_1, \ldots, f_n \in \mathcal{X}^*$. By part (b), there exists $x \in \mathcal{X}$ such that $F(f_i) = f_i(x)$ for all $i \in \{1, \ldots, n\}$ (so far we have not used the independence of $\{f_1, \ldots, f_n\}$). Equivalently

$$|\widehat{f}_i(\widehat{x}) - \widehat{f}_i(F)| = |\widehat{x}(f_i) - F(f_i)| = 0$$

for all $i \in \{1, \ldots, n\}$, in which case $\widehat{x} \in \cap_{i=1}^{n} U_{F\widehat{f}_i\varepsilon_i}$. Sets of this form give a neighbourhood base at F (in the weak* topology), by Theorem 5.14(a). It follows that $F \in \overline{\widehat{\mathcal{X}}}$, so $\widehat{\mathcal{X}}$ is weak*-dense in $\mathcal{X}^{**}$.

If $\|F\| < 1$ we may choose x with $\|x\| < 1$, and if $\|F\| = 1$ then every weak*-neighbourhood U of F contains $G \in \mathcal{X}^{**}$ such that $\|G\| < 1$ (U is also a neighbourhood in the norm topology), and again we may choose x with $\|x\| < 1$. Thus, the closed unit ball of $\widehat{\mathcal{X}}$ is weak*-dense in the closed unit ball of $\mathcal{X}^{**}$.

(e) If $\mathcal{X}$ is reflexive, its closed unit ball corresponds to the closed unit ball of $\mathcal{X}^{**}$, which is weak*-compact by Alaoglu's theorem, and hence weakly compact (in $\mathcal{X}$) by part (c). Conversely, if the closed unit ball of $\mathcal{X}$ is weakly compact, its image in $\mathcal{X}^{**}$ is compact in the (relative) weak* topology, which is Hausdorff by Proposition 5.16, so its image is the closed unit ball of $\mathcal{X}^{**}$ by part (d). Thus, if $F \in \mathcal{X}^{**} \setminus \{0\}$ then $\frac{1}{\|F\|}F = \widehat{x}$ for some $x \in \mathcal{X}$, in which case $F = \widehat{\|F\|x}$. This shows that $\mathcal{X}$ is reflexive.

53. (a) Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathcal{X}$ which converges to $x \in \mathcal{X}$. Then $\langle T_n x\rangle_{n=1}^\infty$ converges to Tx, and by exercise 47(a) there exists $M \in (0,\infty)$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. Given $\varepsilon \in (0,\infty)$, there exists $N \in \mathbb{N}$ such that $\|x_n - x\| < \frac{\varepsilon}{2M}$ and $\|T_n x - Tx\| < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$ with $n \geq N$. It follows that

$$\|T_n x_n - Tx\| \leq \|T_n x_n - T_n x\| + \|T_n x - Tx\| < \|T_n(x_n - x)\| + \frac{\varepsilon}{2} \leq \|T_n\|\|x_n - x\| + \frac{\varepsilon}{2} < M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq N$, which implies that $\langle T_n x_n\rangle_{n=1}^\infty$ converges to Tx.

(b) Let $x \in \mathcal{X}$. Then $\langle S_n x\rangle_{n=1}^\infty$ converges to Sx. By part (a), it follows that $\langle (T_n S_n)x\rangle_{n=1}^\infty = \langle T_n(S_n x)\rangle_{n=1}^\infty$ converges to TSx. This shows that $\langle T_n S_n\rangle_{n=1}^\infty$ converges to TS in the strong operator topology.

56. If $x \in E$, and $y \in E^\perp$, then $x \perp y$ and hence $x \in (E^\perp)^\perp$. It is clear that $(E^\perp)^\perp$ is a closed subspace of $\mathcal{H}$. If F is another closed subspace of $\mathcal{H}$ containing E, then $F^\perp \subseteq E^\perp$ because, for each $x \in F^\perp$, $x \perp y$ for all $y \in F$ and hence all $y \in E$. Similarly $(E^\perp)^\perp \subseteq (F^\perp)^\perp$. Given $x \in (F^\perp)^\perp$, write $x = y + z$ for some $y \in F$ and $z \in F^\perp$ (this works because $\mathcal{H} = F \oplus F^\perp$). Note that $0 = \langle x, z\rangle = \langle y, z\rangle + \langle z, z\rangle = \|z\|^2$, so $z = 0$ and hence $x \in F$. This shows that $(F^\perp)^\perp \subseteq F$, so $(E^\perp)^\perp$ is a subset of every closed subspace of $\mathcal{H}$ containing E, whence it is the smallest one.

57. (a) The definition of T^* is given, and $T^* \in L(\mathcal{H},\mathcal{H})$ because conjugation is an involution. If $x, y \in \mathcal{H}$ then

$$\langle x, T^*y\rangle = \langle x, V^{-1}T^\dagger Vy\rangle = (VV^{-1}T^\dagger Vy)(x) = (Vy)(Tx) = \langle Tx, y\rangle,$$

and if $T' \in L(\mathcal{H},\mathcal{H})$ is another map with this property then $\langle x, T'y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$, so $T'y = T^*y$ for all $y \in \mathcal{H}$ and hence $T' = T^*$.

(b) If $x, y \in \mathcal{H}$ then $\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$, which implies that $T = T^{**}$ by part (a). If $x \in \mathcal{H}$

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leq \|x\|\|T^*Tx\| \leq \|x\|\|T^*\|\|Tx\|$$

and hence $\|Tx\| \leq \|T^*\|\|x\|$, which shows that $\|T\| \leq \|T^*\|$. Thus $\|T^*\| \leq \|T^{**}\| = \|T\|$, so $\|T^*\| = \|T\|$. Now $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$; a variant of the above calculation gives $\|Tx\|^2 \leq \|T^*T\|\|x\|^2$ for all $x \in \mathcal{H}$ and hence $\|T\|^2 \leq \|T^*T\|$. Therefore $\|T^*T\| = \|T\|^2$.

If $a, b \in \mathbb{C}$ and $S, T \in L(\mathcal{H},\mathcal{H})$ then

$$(aS + bT)^* = V^{-1}(aS + bT)^\dagger V = V^{-1}(aS^\dagger + bT^\dagger)V = \overline{a}V^{-1}S^\dagger V + \overline{b}V^{-1}T^\dagger V = \overline{a}S^* + \overline{b}T^*$$

and $(ST)^* = V^{-1}(ST)^\dagger V = V^{-1}T^\dagger S^\dagger V = V^{-1}T^\dagger VV^{-1}S^\dagger V = T^*S^*$.

(c) By Theorem 5.25

$$\begin{aligned}\mathrm{Im}(T)^\perp &= \{x \in \mathcal{H} \mid \langle y, x\rangle = 0 \text{ for all } y \in \mathrm{Im}(T)\}\\ &= \{x \in \mathcal{H} \mid \langle Ty, x\rangle = 0 \text{ for all } y \in \mathcal{H}\}\\ &= \{x \in \mathcal{H} \mid \langle y, T^*x\rangle = 0 \text{ for all } y \in \mathcal{H}\}\end{aligned}$$

$$= \{x \in \mathcal{H} \mid T^*x = 0\}$$
$$= \mathcal{N}(T^*)$$

and hence $\mathcal{N}(T)^\perp = \mathcal{N}(T^{**})^\perp = (\operatorname{Im}(T^*)^\perp)^\perp$ is the smallest closed subspace of $\mathcal{H}$ containing $\operatorname{Im}(T^*)$. In particular $\overline{\operatorname{Im}(T^*)} \subseteq \mathcal{N}(T)^\perp$. But $\overline{\operatorname{Im}(T^*)}$ is a subspace; thus $\overline{\operatorname{Im}(T^*)} = \mathcal{N}(T)^\perp$.

(d) If T is unitary, it is invertible. Moreover $\langle Tx, y\rangle = \langle Tx, TT^{-1}y\rangle = \langle x, T^{-1}y\rangle$ for all $x, y \in \mathcal{H}$, which shows that $T^{-1} = T^*$ by part (a). Conversely, if T is invertible and $T^{-1} = T^*$ then $\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$. In other words T is unitary.

58. (a) Let $x, y \in \mathcal{H}$ and $a \in \mathbb{C}$. Then $(ax + y) - (aPx + Py) = a(x - Px) + (y - Py) \in \mathcal{M}^\perp$ because $\mathcal{M}^\perp$ is a subspace of $\mathcal{H}$, and $aPx + Py \in \mathcal{M}$ because $Px, Py \in \mathcal{M}$ and $\mathcal{M}$ is a subspace of $\mathcal{H}$. Therefore $P(ax + y) = aPx + Py$, which shows that P is linear. If $x \in \mathcal{H}$ then

$$\|Px\|^2 \le \|Px\|^2 + \|x - Px\|^2 = \|Px + x - Px\|^2 = \|x\|^2$$

by Pythagoras, so $\|Px\| \le \|x\|$ and hence $P \in L(\mathcal{H}, \mathcal{H})$. Moreover, if $x, y \in \mathcal{H}$ then

$$\langle Px, y\rangle = \langle Px, y - Py + Py\rangle = \langle Px, y - Py\rangle + \langle Px, Py\rangle = \langle x - x + Px, Py\rangle = \langle x, Py\rangle - \langle x - Px, Py\rangle = \langle x, Py\rangle,$$

so $P^* = P$. If $x \in \mathcal{H}$ then $P^2x = P(Px) \in \mathcal{M}$ and $x - P^2x = (x - Px) + (Px - P^2x) \in \mathcal{M}^\perp$, which implies that $Px = P^2x$. Therefore $P^2 = P$. If $x \in \mathcal{M}$ then $x - x = 0 \in \mathcal{M}^\perp$, so $Px = x$ and hence $x \in \operatorname{Im}(P)$. Since $\operatorname{Im}(P) \subseteq \mathcal{M}$ by definition, it follows that $\operatorname{Im}(P) = \mathcal{M}$. Therefore $\mathcal{M}^\perp = \operatorname{Im}(P)^\perp = \mathcal{N}(P^*) = \mathcal{N}(P)$.

(b) Since $P^2 = P$ it is clear that $\operatorname{Im}(P) = \mathcal{N}(P - I)$, which is closed because $P - I$ is bounded. If $x \in \mathcal{H}$ then $x - Px \in \mathcal{N}(P) = \mathcal{N}(P^*) = \operatorname{Im}(P)^\perp$ because $P^2 = P$. Therefore P is the orthogonal projection onto $\operatorname{Im}(P)$.

(c) Let $x \in \mathcal{H}$. Clearly $\sum_{\alpha\in A}\langle x, u_\alpha\rangle u_\alpha \in \mathcal{M}$. If $a_\alpha \in \mathbb{C}$ for each $\alpha \in A$ then

$$\left\langle x - \sum_{\alpha\in A}\langle x, u_\alpha\rangle u_\alpha, \sum_{\alpha\in A} a_\alpha u_\alpha\right\rangle = \sum_{\alpha\in A}\overline{a}_\alpha\left\langle x - \sum_{\beta\in A}\langle x, u_\beta\rangle u_\beta, u_\alpha\right\rangle = \sum_{\alpha\in A}\overline{a}_\alpha(\langle x, u_\alpha\rangle - \langle x, u_\alpha\rangle) = 0,$$

which implies that $x - \sum_{\alpha\in A}\langle x, u_\alpha\rangle u_\alpha \in \mathcal{M}^\perp$. Therefore $Px = \sum_{\alpha\in A}\langle x, u_\alpha\rangle u_\alpha$.

59. Choose a sequence $\langle x_n\rangle_{n=1}^\infty$ in K such that $\langle\|x_n\|\rangle_{n=1}^\infty$ converges to $r := \inf_{x\in K}\|x\|$. Given $\varepsilon \in (0, \infty)$, there exists $N \in \mathbb{N}$ such that $\|x_n\| < r + \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$ with $n \ge N$. If $m, n \in \mathbb{N}$ and $m \ge n \ge N$ then

$$\|x_m - x_n\|^2 + \|x_m + x_n\|^2 = \langle x_m - x_n, x_m - x_n\rangle + \langle x_m + x_n, x_m + x_n\rangle = 2\|x_m\|^2 + 2\|x_n\|^2 < 4(r + \tfrac{\varepsilon}{2})^2$$

and $\|x_m + x_n\|^2 = 4\|\frac{1}{2}x_m + \frac{1}{2}x_n\|^2 \ge 4r^2$ (because $\frac{1}{2}x_m + \frac{1}{2}x_n \in K$), so $\|x_m - x_n\|^2 < 4\frac{\varepsilon^2}{4} = \varepsilon^2$ and hence $\|x_m - x_n\| < \varepsilon$. This shows that $\langle x_n\rangle_{n=1}^\infty$ is a Cauchy sequence, so it has a limit $x \in \overline{K} = K$. Since $\|\cdot\|$ is continuous, it is clear that $\|x\| = \lim_{n\to\infty}\|x_n\| = r$, so x has the smallest norm among the elements of K. It is clear that x is unique, because if $y \in K$ and $\|y\| = r$ then

$$\|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = 2r^2 + 2r^2 - 4\|\tfrac{1}{2}x + \tfrac{1}{2}y\|^2 \le 4r^2 - 4r^2 = 0.$$

63. (a) Let $\langle u_n\rangle_{n=1}^\infty$ be an orthonormal sequence in $\mathcal{H}$. If $f \in \mathcal{H}^*$, there exists $x \in \mathcal{H}$ such that $f = \langle\cdot, x\rangle$. Also

$$\sum_{n=1}^\infty |\langle x, u_n\rangle|^2 \le \|x\|^2$$

(this is Bessel's inequality), so $\lim_{n\to\infty}|f(u_n)|^2 = \lim_{n\to\infty}|\langle u_n, x\rangle|^2 = \lim_{n\to\infty}|\langle x, u_n\rangle|^2 = 0$. It follows that $\lim_{n\to\infty} f(u_n) = 0 = f(0)$, so $\langle u_n\rangle_{n=1}^\infty$ converges weakly to 0.

(b) Let $x \in B$. If $x \neq 0$, construct an orthonormal sequence $\{u_n\}_{n=0}^\infty$ in $\mathcal{H}$ such that $u_0 = \frac{x}{\|x\|}$, using the Gram-Schmidt process and the fact that $\mathcal{H}$ is infinite-dimensional. Otherwise, construct an orthonormal sequence $\{u_n\}_{n=1}^\infty$ in $\mathcal{H}$. In either case $u_n \perp x$ for all $n \in \mathbb{N}$. Set $a := \sqrt{1 - \|x\|^2}$, and for each $n \in \mathbb{N}$ define $x_n := au_n + x$. Then $(x_m - x) \perp (x_n - x)$ for all $m, n \in \mathbb{N}$, and

$$\|x_n\|^2 = \langle au_n + x, au_n + x\rangle = |a|^2\|u_n\|^2 + a\langle u_n, x\rangle + \overline{a}\langle x, u_n\rangle + \|x\|^2 = a^2 + 0 + 0 + \|x\|^2 = 1$$

for all $n \in \mathbb{N}$. This implies that $\langle x_n\rangle_{n=1}^\infty$ is a sequence in S. Moreover, by Bessel's inequality

$$\sum_{n=1}^\infty |\langle y, x_n - x\rangle|^2 \leq \sum_{n=1}^\infty \|x_n - x\|^2 |\langle y, \tfrac{x_n - x}{\|x_n - x\|}\rangle|^2 \leq \sum_{n=1}^\infty (\|x_n\| + \|x\|)^2 |\langle y, \tfrac{x_n - x}{\|x_n - x\|}\rangle|^2 \leq 4\|y\|^2,$$

and hence $\lim_{n\to\infty}\langle x_n - x, y\rangle = 0$, for all $y \in \mathcal{H}$. This implies that $\langle x_n\rangle_{n=1}^\infty$ converges weakly to x, by a similar argument to part (a). Therefore x is in the weak closure of S, so S is weakly dense in B.

66. (a) Let $T : \mathcal{M} \hookrightarrow C([0,1])$ be the inclusion. Also let $\langle f_n\rangle_{n=1}^\infty$ be a sequence in $\mathcal{M}$ which converges to $f \in \mathcal{M}$ such that $\langle Tf_n\rangle_{n=1}^\infty$ converges to $g \in C([0,1])$. Suppose that $Tf \neq g$. Then $|f(x) - g(x)| > 0$ for some $x \in [0,1]$, so there exist $\varepsilon, \delta \in (0,\infty)$ such that $|f(y) - g(y)| > \varepsilon$ for all $y \in [0,1]$ such that $|x - y| < \delta$. Moreover, there exists $n \in \mathbb{N}$ such that $\|f - f_n\|_{L^2} < \frac{\varepsilon\sqrt{\delta}}{2}$ and $\|Tf_n - g\|_u < \frac{\varepsilon\sqrt{\delta}}{2}$. It follows that

$$\varepsilon\sqrt{\delta} \leq \sqrt{\int_0^1 |f - g|^2} = \|f - g\|_{L^2} \leq \|f - f_n\|_{L^2} + \|Tf_n - g\|_{L^2} < \frac{\varepsilon\sqrt{\delta}}{2} + \sqrt{\int_0^1 \|Tf_n - g\|_u^2} < \varepsilon\sqrt{\delta},$$

which is plainly impossible. Therefore $Tf = g$, which shows that T is a closed linear map, so by the closed graph theorem (note that $\mathcal{M}$ is closed, thus complete, and $C([0,1])$ is also complete) there exists $C \in (0,\infty)$ such that $\|f\|_u = \|Tf\|_u \leq C\|f\|_{L^2}$ for all $f \in \mathcal{M}$.

(b) Given $x \in [0,1]$, define $\widehat{x} : \mathcal{M} \to \mathbb{C}$ by $\widehat{x}(f) := f(x)$. Since addition and scalar multiplication in $\mathcal{M}$ are defined pointwise, $\widehat{x}$ is linear. Moreover, if $f \in \mathcal{M}$ then $|\widehat{x}(f)| = |f(x)| \leq \|f\|_u \leq C\|f\|_{L^2}$, so $\widehat{x} \in \mathcal{M}^*$. It follows that $\widehat{x} = \langle \cdot, g_x\rangle$ for some $g_x \in \mathcal{M}$. In other words, $f(x) = \langle f, g_x\rangle$ for all $f \in \mathcal{M}$. In particular, $\|g_x\|_{L^2}^2 = \langle g_x, g_x\rangle = g_x(x) \leq \|g_x\|_u \leq C\|g_x\|_{L^2}$, which implies that $\|g_x\|_{L^2} \leq C$ (even if $\|g_x\|_{L^2} = 0$).

(c) Let $\langle f_n\rangle_{n=1}^N$ be a finite orthonormal sequence in $\mathcal{M}$. If $x \in [0,1]$ there exists $g_x \in \mathcal{M}$ such that $f(x) = \langle f, g_x\rangle$ for all $f \in \mathcal{M}$. Using Bessel's inequality, it follows that

$$\sum_{n=1}^N |f_n(x)|^2 = \sum_{n=1}^N |\langle f_n, g_x\rangle|^2 = \sum_{n=1}^N |\langle g_x, f_n\rangle|^2 \leq \|g_x\|_{L^2}^2 \leq C^2.$$

Therefore

$$N = \sum_{n=1}^N \|f_n\|_{L^2}^2 = \sum_{n=1}^N \int_0^1 |f_n|^2 = \int_0^1 \sum_{n=1}^N |f_n|^2 \leq \int_0^1 C^2 = C^2,$$

so every linearly independent subset of $\mathcal{M}$ has cardinality at most C^2 (otherwise one could construct an orthonormal set with more than C^2 elements using the Gram-Schmidt process).

67. Note that $\mathcal{M} = \mathcal{N}(U - I) = \mathcal{N}(U(I - U^*)) = \mathcal{N}(I - U^*) = \mathcal{N}((I - U)^*) = \operatorname{Im}(I - U)^\perp$ is a closed subspace of $\mathcal{H}$, by Exercise 57. Therefore $\mathcal{M}^\perp = (\operatorname{Im}(I - U)^\perp)^\perp = \overline{\operatorname{Im}(I - U)}$ by Exercise 56. If $x \in \mathcal{M}$ and $n \in \mathbb{N}$ then

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} U^k x = \frac{1}{n}\sum_{k=0}^{n-1} x = \frac{n}{n}x = x,$$

and hence $\lim_{n\to\infty} S_n x = x$. On the other hand, if $x \in \mathcal{M}^\perp$ and $\varepsilon \in (0,\infty)$, there exists $y \in \operatorname{Im}(I-U)$ such that $\|x-y\| < \frac{\varepsilon}{2}$. Moreover, if $z \in \mathcal{H}$ such that $y = z - Uz$ then

$$\|S_n y\| = \|S_n(z-Uz)\| = \left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k z - \frac{1}{n}\sum_{k=0}^{n-1} U^{k+1} z\right\| = \frac{1}{n}\|z - U^n z\| \leq \frac{1}{n}(\|z\| + \|U^n z\|) = \frac{2}{n}\|z\|$$

for all $n \in \mathbb{N}$, so there exists $N \in \mathbb{N}$ such that $\|S_n y\| < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$ with $n \geq N$. It follows that

$$\|S_n x\| \leq \|S_n(x-y)\| + \|S_n y\| < \left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k(x-y)\right\| + \frac{\varepsilon}{2} \leq \frac{1}{n}\sum_{k=0}^{n-1}\|U^k(x-y)\| + \frac{\varepsilon}{2} = \frac{n}{n}\|x-y\| + \frac{\varepsilon}{2} < \varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq N$. Therefore $\lim_{n\to\infty} S_n x = 0$. If $x \in \mathcal{H}$ then $Px \in \mathcal{M}$ and $x - Px \in \mathcal{M}^\perp$, so

$$\lim_{n\to\infty} S_n x = \lim_{n\to\infty} S_n(Px + x - Px) = \lim_{n\to\infty} S_n(Px) + \lim_{n\to\infty} S_n(x - Px) = Px.$$

This shows that $\langle S_n \rangle_{n=1}^\infty$ converges to P in the strong operator topology.