3. Since L^p and L^r are subspaces of $\mathbb{C}^X$, their intersection is a vector space. It is clear that $\|\cdot\|$ is a norm (this follows directly from the fact that $\|\cdot\|_p$ and $\|\cdot\|_r$ are norms). Let $\langle f_n\rangle_{n=1}^\infty$ be a Cauchy sequence in $L^p \cap L^r$. Since $\|f_m - f_n\|_p \leq \|f_m - f_n\|$ and $\|f_m - f_n\|_r \leq \|f_m - f_n\|$ for all $m, n \in \mathbb{N}$, it is clear that $\langle f_n\rangle_{n=1}^\infty$ is a Cauchy sequence in both L^p and L^r. Let $g_p \in L^p$ and $g_r \in L^r$ be the respective limits of this sequence. Given $\varepsilon \in (0,\infty)$, there exists $N \in \mathbb{N}$ such that $\|f_n - g_p\|_p < \varepsilon^{(p+1)/p}$ for all $n \in \mathbb{N}$ with $n \geq N$. If $n \in \mathbb{N}$ and $n \geq N$

$$\mu(E)\varepsilon^p = \int_E \varepsilon^p \leq \int_E |f_n - g_p|^p \leq \int |f_n - g_p|^p < \varepsilon^{(p+1)p/p} = \varepsilon^{p+1},$$

so $\mu(E) < \varepsilon$, where $E := \{x \in X \mid \varepsilon^p \leq |f_n(x) - g_p(x)|^p\} = \{x \in X \mid \varepsilon \leq |f_n(x) - g_p(x)|\}$. This shows that $\langle f_n\rangle_{n=1}^\infty$ converges in measure to g_p. If $r < \infty$, a similar argument shows that $\langle f_n\rangle_{n=1}^\infty$ converges in measure to g_r. Otherwise, some subsequence of $\langle f_n\rangle_{n=1}^\infty$ converges to g_p almost everywhere, and this subsequence also converges to g_r almost everywhere (since $\langle f_n\rangle_{n=1}^\infty$ converges uniformly to g_r off a set of measure 0). In either case $g_p = g_r$, so $g_p \in L^p \cap L^r$. Since $\langle f_n\rangle_{n=1}^\infty$ converges to g_p in L^p and L^r, it easily follows that $\langle f_n\rangle_{n=1}^\infty$ converges to g_p in $L^p \cap L^r$. This shows that $L^p \cap L^r$ is a Banach space.

Now let $q \in (p, r)$. There exists $\lambda \in (0,1)$ such that $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ for all $f \in L^p \cap L^r$. In particular, if $f \in L^p \cap L^r$ and $\|f\| = 1$, then $\|f\|_p \leq 1$ and $\|f\|_r \leq 1$ and hence $\|f\|_q \leq 1^\lambda 1^{1-\lambda} = 1$. This shows that the linear map $L^p \cap L^r \hookrightarrow L^q$ is continuous.

4. Since L^p and L^r are subspaces of $\mathbb{C}^X$, their sum is a vector space. Clearly $\|f\| \geq 0$ for all $f \in L^p + L^r$. Let $f \in L^p + L^r$ and $a \in \mathbb{C}$. If $a = 0$ then $\|af\| = \|0\| = 0 = |a|\|f\|$ because

$$0 = \|0\|_p + \|0\|_r \in \{\|g\|_p + \|h\|_r \mid g \in L^p, h \in L^r \text{ and } f = g + h\}.$$

Otherwise $\|af\| \leq \|ag\|_p + \|ah\|_r = |a|(\|g\|_p + \|h\|_r)$ for all $g \in L^p$ and $h \in L^r$ with $f = g + h$, and hence $\|af\|/|a| \leq \|f\|$. This implies that $|a|\|f\| = \|a^{-1}af\|/|a^{-1}| \leq \|af\|$, which shows that $\|af\| = |a|\|f\|$. Now let $f_1, f_2 \in L^p + L^r$ and $\varepsilon \in (0,\infty)$. There exist $g_1, g_2 \in L^p$ and $h_1, h_2 \in L^r$ such that $f_1 = g_1 + h_1$, $f_2 = g_2 + h_2$, $\|g_1\|_p + \|h_1\|_r < \|f_1\| + \varepsilon$ and $\|g_2\|_p + \|h_2\|_r < \|f_2\| + \varepsilon$. It follows that

$$\|f_1 + f_2\| \leq \|g_1 + g_2\|_p + \|h_1 + h_2\|_r \leq \|g_1\|_p + \|g_2\|_p + \|h_1\|_r + \|h_2\|_r < \|f_1\| + \|f_2\| + 2\varepsilon,$$

which shows that $\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$. Finally, let $f \in L^p + L^r$, and suppose that $\|f\| = 0$. Given $\varepsilon \in (0, 1)$, there exist $g \in L^p$ and $h \in L^r$ such that $f = g + h$ and $\|g\|_p + \|h\|_r < \varepsilon^{(p+1)/p}$. It follows that $\mu(G) < \varepsilon$, where $G := \{x \in X \mid \varepsilon \leq |g(x)|\}$, because

$$\mu(G)\varepsilon^p = \int_G \varepsilon^p \leq \int_G |g|^p \leq \int |g|^p < \varepsilon^{(p+1)p/p} = \varepsilon^{p+1}.$$

Similarly $\mu(H) < \varepsilon$ where $H := \{x \in X \mid \varepsilon \leq |h(x)|\}$ (if $r = \infty$ then $\mu(H) = 0$ because $\varepsilon^{(p+1)/p} < \varepsilon$). Define $F := \{x \in X \mid 2\varepsilon \leq |f(x)|\}$, so that $F \subseteq G \cup H$. Then $\mu(F) \leq \mu(G \cup H) < 2\varepsilon$. It follows that

$$\{x \in X \mid 0 < |f(x)|\} = \cup_{n=1}^\infty \cap_{k=n}^\infty \{x \in X \mid 2k^{-1} \leq |f(x)|\},$$

has measure 0, so $f = 0$ almost everywhere. This shows that $\|\cdot\|$ is a norm.

Let $\sum_{n=1}^\infty f_n$ be an absolutelty convergent series in $L^p + L^r$. For each $n \in \mathbb{N}$ there exist $g_n \in L^p$ and $h_n \in L^r$ such that $f_n = g_n + h_n$ and $\|g_n\|_p + \|h_n\|_r < \|f_n\| + 2^{-n}$. It follows that $\sum_{n=1}^\infty g_n$ and $\sum_{n=1}^\infty h_n$ are absolutely convergent series in L^p and L^r respectively, so they have limits $g \in L^p$ and $h \in L^r$ respectively. This convergence also holds in

$L^p + L^r$, because $\|\sum_{n=1}^N g_n - g\| \le \|\sum_{n=1}^N g_n - g\|_p + \|0\|_r$ and $\|\sum_{n=1}^N h_n - h\| \le \|0\|_p + \|\sum_{n=1}^N h_n - h\|_r$ for all $N \in \mathbb{N}$. Therefore $\sum_{n=1}^\infty f_n = \sum_{n=1}^\infty (g_n + h_n)$ has a limit in $L^p + L^r$, namely $g + h$. This shows that $L^p + L^r$ is a Banach space.

Now let $q \in (p, r)$ and let $f \in L^q$ with $\|f\|_q = 1$. Define $E \coloneqq \{x \in X \mid 1 < |f(x)|\}$, $g \coloneqq f\chi_E$ and $h \coloneqq f\chi_{E^c}$. Then $|g|^p \le |g|^q \le |f|^q$ and $|h|^r \le |h|^q \le |f|^q$, so $g \in L^p$ and $h \in L^r$, which implies that

$$\|f\| = \|g + h\| \le \|g\|_p + \|h\|_r \le \left(\int |f|^q\right)^{1/p} + \left(\int |f|^q\right)^{1/r} = 1^{1/p} + 1^{1/p} = 2.$$

This shows that the linear map $L^q \hookrightarrow L^p + L^r$ is continuous.

6. Most of the hard work has been done as part of exercise 2.64.

   (a) If $p_1 < \infty$, define $f(x) \coloneqq x^{-1/p_1}\chi_{(0,\frac{1}{2})}(x) + x^{-1/p_0}\chi_{(2,\infty)}(x)$. Otherwise replace x^{-1/p_1} by $|\log(x)|$.

   (b) If $p_1 < \infty$, set $f(x) \coloneqq x^{-1/p_1}|\log(x)|^{-2/p_1}\chi_{(0,\frac{1}{2})} + x^{-1/p_0}|\log(x)|^{-2/p_0}\chi_{(2,\infty)}$. Otherwise, omit the first term.

   (c) Set $p_1 \coloneqq p_0$ in part (b).

9. In exercise 3 we showed that convergence in L^p implies convergence in measure (which in turn implies Cauchyness in measure, hence subsequential convergence almost everywhere). Conversely, let $\langle f_n\rangle_{n=1}^\infty$ be a sequence of measurable functions in $\mathbb{C}^X$ which converges to $f : X \to \mathbb{C}$ in measure. Suppose there exists $g \in L^p$ such that $|f_n| \le g$ for all $n \in \mathbb{N}$. If $\langle f_{n_k}\rangle_{k=1}^\infty$ is a subsequence of $\langle f_n\rangle_{n=1}^\infty$, it converges to f in measure, so it has a further subsequence $\langle f_{n_{k_j}}\rangle_{j=1}^\infty$ which converges to f almost everywhere. It follows that $\langle |f_{n_{k_j}}|^p\rangle_{j=1}^\infty$ converges to $|f|^p$ almost everywhere, so by the dominated convergence theorem $|f|^p \in L^1$, whence $f \in L^p$ (note that $|f_n|^p \le g^p$ for all $n \in \mathbb{N}$, and $g^p \in L^1$). Furthermore $\langle |f_{n_{k_j}} - f|^p\rangle_{j=1}^\infty$ converges to 0 almost everywhere, and since $|f_{n_{k_j}} - f|^p \le 2^p(g^p + |f|^p)$ for all $j \in \mathbb{N}$, the dominated convergence theorem implies that $\langle \|f_{n_{k_j}} - f\|_p^p\rangle_{j=1}^\infty$ converges to 0, in which case $\langle \|f_{n_{k_j}} - f\|_p\rangle_{j=1}^\infty$ also converges to 0. This shows that every subsequence of $\langle \|f_n - f\|_p\rangle_{n=1}^\infty$ has a subsequence which converges to 0, so $\langle \|f_n - f\|_p\rangle_{n=1}^\infty$ itself converges to 0.

10. Let $\langle f_n\rangle_{n=1}^\infty$ be a sequence in L^p which converges to $f \in L^p$ almost everywhere. Suppose that $\langle \|f_n\|_p\rangle_{n=1}^\infty$ converges to $\|f\|_p$. Since $|f_n - f|^p \le 2^p(|f_n|^p + |f|^p)$, $\langle |f_n - f|^p\rangle_{n=1}^\infty$ and $\langle 2^p(|f_n|^p + |f|^p)\rangle_{n=1}^\infty$ converge to 0 and $2^{p+1}|f|^p$ almost everywhere, and

$$\lim_{n\to\infty} \int 2^p(|f_n|^p + |f|^p) = 2^p \lim_{n\to\infty} \|f_n\|_p^p + 2^p\|f\|_p^p = 2^p\left(\lim_{n\to\infty} \|f_n\|_p\right)^p + 2^p\|f\|_p^p = 2^p\|f\|_p^p + 2^p\|f\|_p^p = \int 2^{p+1}|f|^p,$$

the generalised dominated convergence theorem implies that $\langle \|f_n - f\|_p^p\rangle_{n=1}^\infty$ converges to 0, in which case $\langle \|f_n - f\|_p\rangle_{n=1}^\infty$ also converges to 0. The converse follows from the reverse triangle inequality.

13. Let $f \in L^p(\mathbb{R}^n, m)$ and $\varepsilon \in (0, \infty)$. By Proposition 6.7 there is a simple function $g = \sum_{j=1}^r a_j\chi_{E_j}$ such that $m(\cup_{j=1}^r E_j) < \infty$ and $\|f - g\|_p < \frac{\varepsilon}{4}$. Fix $j \in \{1, \dots, r\}$ and assume without loss of generality that $m(E_j) > 0$. There exists $b_j \in \mathbb{Q}(i)$ such that $\|a_j\chi_{E_j} - b_j\chi_{E_j}\|_p = |a_j - b_j|m(E_j)^{1/p} < \frac{\varepsilon}{4r}$. Moreover, by Theorem 2.40(c) there is a finite collection of rectangles whose sides are intervals such that $\|b_j\chi_{E_j} - b_j\chi_{F_j}\|_p = |b_j|m(E_j \triangle F_j)^{1/p} < \frac{\varepsilon}{4r}$, where F_j is the union of these rectangles. We may shrink F_j to give a finite collection of rectangles with sides that are intervals with rational endpoints, whose union G_j satisfies $\|b_j\chi_{F_j} - b_j\chi_{G_j}\|_p < \frac{\varepsilon}{4r}$. It follows that

$$\left\|f - \sum_{j=1}^r b_j\chi_{G_j}\right\|_p \le \|f - g\| + \sum_{j=1}^r \|a_i\chi_{E_i} - b_i\chi_{E_i}\|_p + \sum_{j=1}^r \|b_i\chi_{E_i} - b_i\chi_{F_i}\|_p + \sum_{j=1}^r \|b_i\chi_{F_i} - b_i\chi_{G_i}\|_p < \varepsilon,$$

so the collection of rational linear combinations of characteristic functions of finite unions of rectangles with sides that are intervals with rational endpoints (which is countable) is dense in $L^p(\mathbb{R}^n, m)$.

For each $r \in (0, \infty)$ set $f_r := \chi_{B_r(0)}$, and note that $\|f_r - f_s\|_\infty = 1$ for all $r, s \in (0, \infty)$ such that $r \neq s$ (find some open rectangle inside $B_r(0) \triangle B_s(0)$). If $D \subseteq L^\infty(\mathbb{R}^n, m)$ is dense then $B_{1/2}(f_r) \cap D \neq \varnothing$ for all $r \in (0, \infty)$, so there is an injective function $(0, \infty) \to D$. In particular D is uncountable.

15. Let $\langle f_n \rangle_{n=1}^\infty$ be a Cauchy sequence in L^p. Let $\varepsilon, \delta \in (0, \infty)$. There exists $N \in \mathbb{N}$ such that $\|f_m - f_n\|_p < \varepsilon\delta^{1/p}$ for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$. If $m, n \in \mathbb{N}$ and $m \geq n \geq N$ then

$$\mu(E)\varepsilon^p = \int_E \varepsilon^p \leq \int_E |f_m - f_n|^p \leq \int |f_m - f_n|^p = \|f_m - f_n\|_p^p < (\varepsilon\delta)^{(p+1)p/p} = \varepsilon^p\delta,$$

so $\mu(E) < \delta$, where $E := \{x \in X \mid \varepsilon \leq |f_m(x) - f_n(x)|\} = \{x \in X \mid \varepsilon^p \leq |f_m(x) - f_n(x)|^p\}$. This shows that $\langle f_n \rangle_{n=1}^\infty$ is Cauchy in measure. Now let $\varepsilon \in (0, \infty)$. There exists $N \in \mathbb{N}$ such that $\|f_m - f_n\|_p < 2^{-(p+1)/p}\varepsilon^{1/p}$ for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$. Since $\{|f_n|^p\}_{n=1}^N$ is a finite subset of L^1, it is uniformly integrable by exercise 11 in section 3.2. Hence, there exists $\delta \in (0, \infty)$ such that $|\int_E |f_n|^p| < 2^{-(p+1)}\varepsilon < \varepsilon$ for all $n \in \mathbb{N}$ and $E \in \mathcal{M}$ with $n \leq N$ and $\mu(E) < \delta$. If $n \in \mathbb{N}$ and $E \in \mathcal{M}$ such that $n > N$ and $\mu(E) < \delta$, then

$$\left|\int_E |f_n|^p\right| = \int_E |f_n|^p \leq \int_E 2^p(|f_n - f_N|^p + |f_N|^p) \leq 2^p\|f_n - f_N\|_p^p + 2^p\left|\int_E |f_N|^p\right| < 2^{-1}\varepsilon + 2^{-1}\varepsilon = \varepsilon.$$

This shows that $\{|f_n|^p\}_{n=1}^\infty$ is uniformly integrable. Now fix $n \in \mathbb{N}$, set $F := \{x \in X \mid 0 < |f_n(x)|^p\}$ and define $F_m := \{x \in X \mid m^{-1} \leq |f_n(x)|^p\}$ for each $m \in \mathbb{N}$. Then $\langle F_m \rangle_{m=1}^\infty$ is increasing and $\cup_{m=1}^\infty F_m = F$, so

$$\lim_{m\to\infty} \int_{F_m^c} |f_n|^p = \lim_{m\to\infty} \left(\int |f_n|^p - \int_{F_m} |f_n|^p\right) = \int |f_n|^p - \lim_{m\to\infty} \int_{F_m} |f_n|^p = \int |f_n|^p - \int_F |f_n|^p = 0.$$

Hence, there exists $m \in \mathbb{N}$ such that $\int_{F_m^c} |f_n|^p < 2^{-(p+1)}\varepsilon$. Define $E_n := F_m$, and note that $\mu(E_n) < \infty$ because

$$\mu(E_n)m^{-1} = \int_{E_n} m^{-1} \leq \int_{E_n} |f_n|^p \leq \int |f_n|^p < \infty.$$

Therefore $\mu(E) < \infty$, where $E := \cup_{n=1}^N E_n$. Clearly $\int_{E^c} |f_n|^p \leq \int_{E_n^c} |f_n|^p < 2^{-(p+1)}\varepsilon < \varepsilon$ for all $n \in \mathbb{N}$ with $n \leq N$. If $n \in \mathbb{N}$ and $n > N$ then

$$\int_{E^c} |f_n|^p \leq \int_{E^c} 2^p(|f_n - f_N|^p + |f_N|^p) \leq 2^p\|f_n - f_N\|_p^p + 2^p\int_{E^c} |f_N|^p < 2^{-1}\varepsilon + 2^{-1}\varepsilon = \varepsilon.$$

This shows that $\langle f_n \rangle_{n=1}^\infty$ has property (iii).

Conversely, let $\langle f_n \rangle_{n=1}^\infty$ be a sequence in L^p which satisfies properties (i)-(iii). Let $\varepsilon \in (0, \infty)$ and choose $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p < 2^{-(p+3)}\varepsilon^p$ for all $n \in \mathbb{N}$. If $\mu(E) = 0$ then

$$\int |f_m - f_n|^p = \int_{E^c} |f_m - f_n|^p \leq \int_{E^c} 2^p(|f_m|^p + |f_n|^p) = 2^p\int_{E^c} |f_m|^p + 2^p\int_{E^c} |f_n|^p < 2^{p+1-(p+3)}\varepsilon^p = \frac{\varepsilon^p}{4} < \varepsilon^p$$

and hence $\|f_m - f_n\|_p < \varepsilon$ for all $m, n \in \mathbb{N}$. Otherwise, take $\delta \in (0, \infty)$ such that $\int_F |f_n|^p = |\int_F |f_n|^p| < 2^{-(p+3)}\varepsilon^p$ for all $n \in \mathbb{N}$ and $F \in \mathcal{M}$ with $\mu(F) < \delta$. For each $m, n \in \mathbb{N}$ define

$$A_{mn} := \{x \in X \mid \varepsilon(2\mu(E))^{-1/p} \leq |f_m(x) - f_n(x)|\},$$

so that

$$\int_{E\setminus A_{mn}} |f_m - f_n|^p \leq \int_{E\setminus A_{mn}} \varepsilon^p(2\mu(E))^{-1} \leq \int_E \varepsilon^p(2\mu(E))^{-1} = \frac{\varepsilon^p}{2}.$$

There exists $N \in \mathbb{N}$ such that $\mu(A_{mn}) < \delta$ for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$, in which case

$$\int_{A_{mn}} |f_m - f_n|^p \leq \int_{A_{mn}} 2^p(|f_m|^p + |f_n|^p) \leq 2^p \int_{A_{mn}} |f_m|^p + 2^p \int_{A_{mn}} |f_n|^p < 2^{p+1-(p+3)}\varepsilon^p = \frac{\varepsilon^p}{4}.$$

It follows that

$$\int |f_m - f_n|^p \leq \int_{E^c} |f_m - f_n|^p + \int_{E\setminus A_{mn}} |f_m - f_n|^p + \int_{A_{mn}} |f_m - f_n|^p < \frac{\varepsilon^p}{4} + \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{4} = \varepsilon^p,$$

and hence $\|f_m - f_n\|_p < \varepsilon$, for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$. This shows that $\langle f_n\rangle_{n=1}^\infty$ is Cauchy in L^p.

18. (a) Let $f \in L^2(\lambda)$ and suppose that $\|f\|_2 = 1$. For every simple function $\phi : X \to [0, \infty]$, hence every $\phi \in L^+$ and in fact every $\phi \in L^1(\lambda)$, it is clear that $\int \phi\, d\lambda = \int \phi\, d\mu + \int \phi\, d\nu$. Therefore

$$\left|\int f\, d\nu\right| \leq \int |f|\, d\nu \leq \int |f|\, d\mu + \int |f|\, d\nu = \int |f|\, d\lambda \leq \|f\|_2\|1\|_2 = \sqrt{\lambda(X)}$$

by Hölder's inequality. This shows that the linear functional $f \mapsto \int f\, d\nu$ on $L^2(\lambda)$ is bounded, so there exists $g \in L^2(\lambda)$ such that $\int f\, d\nu = \int fg\, d\lambda = \int fg\, d\mu + \int fg\, d\nu$, and hence $\int f(1-g)\, d\nu = \int fg\, d\mu$, for all $f \in L^2(\lambda)$. In fact $\int_E f(1-g)\, d\nu = \int_E fg\, d\mu$ for all $f \in L^2(\lambda)$ and $E \in \mathcal{M}$ (as $f\chi_E \in L^2(\lambda)$).

(b) Note that $\int_E g\, d\lambda = \int \chi_E g\, d\lambda = \int \chi_E\, d\nu = \nu(E) \geq 0$ and $\int_E (1-g)\, d\lambda = \int_E d\lambda - \int_E g\, d\lambda = \lambda(E) - \nu(E) \geq 0$ for all $E \in \mathcal{M}$. In particular $\int_E \mathrm{Im}(g)\, d\lambda = 0$ for all $E \in \mathcal{M}$, which implies that $\mathrm{Im}(g) = 0$ λ-almost everywhere. Moreover, $\lambda(\{x \in X \mid g(x) \leq -n^{-1}\}) = 0$ for all $n \in \mathbb{N}$, which implies that $g \geq 0$ λ-almost everywhere. Similarly $1 - g \geq 0$, whence $0 \leq g \leq 1$, λ-almost everywhere. In particular $0 \leq g \leq 1$ μ-almost everywhere and ν-almost everywhere.

(c) Clearly $\nu_s(E) = \nu(\varnothing) = 0$ for all $E \in \mathcal{M}$ with $E \subseteq B^c$. Moreover

$$\mu(E) = \int \chi_E\, d\mu = \int \chi_E \cdot g\, d\mu = \int \chi_E(1-g)\, d\nu = \int 0\, d\nu = 0$$

for all $E \in \mathcal{M}$ with $E \subseteq B$. Therefore $\nu_s \perp \mu$. Since $(1-g)^{-1}\chi_A \geq 0$ ν-almost everywhere, there is an increasing sequence of simple functions $\langle\phi_n\rangle_{n=1}^\infty$ which converges to $(1-g)^{-1}\chi_A$ ν-almost everywhere. Since $(1-g)^{-1}\chi_A < \infty$ pointwise, $\phi_n \in L^2(\lambda)$ for all $n \in \mathbb{N}$. Therefore

$$\nu_a(E) = \nu(A \cap E) = \int_E \frac{\chi_A}{1-g}(1-g)\, d\nu = \lim_{n\to\infty} \int_E \phi_n(1-g)\, d\nu = \lim_{n\to\infty} \int_E \phi_n g\, d\mu = \int_E g(1-g)^{-1}\chi_A\, d\mu$$

for all $E \in \mathcal{M}$, by the monotone convergence theorem. This shows that $\nu_a \ll \mu$.

19. Note that $\phi_n = \phi_{n^{-1}\chi_{\{1,\ldots,n\}}}$ and hence $\|\phi_n\| = \|n^{-1}\chi_{\{1,\ldots,n\}}\|_1 = 1$, by Proposition 6.13. It follows by Alaoglu's theorem and Theorem 4.29 that $(\phi_n)_{n=1}^\infty$ has a weak* cluster point $\phi \in (l^\infty)^*$. Suppose that $\phi = \phi_g$ for some $g \in l^1$. If $k \in \mathbb{N}$ and $\varepsilon \in (0, \infty)$, then $\phi(\chi_{\{k\}}) = g(k)$ and hence $(\phi_n)_{n=1}^\infty$ is frequently in $\widehat{\chi_{\{k\}}}^{-1}(B_{\varepsilon/2}(g(k)))$. In particular, there exists $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{\varepsilon}{2}$ and $n \geq k$ such that $|g(k) - \frac{1}{n}| = |g(k) - \phi_n(\chi_{\{k\}})| < \frac{\varepsilon}{2}$. It follows that $g(k) < \varepsilon$, so $g = 0$. Therefore $(\phi_n)_{n=1}^\infty$ is frequently in $\widehat{\chi_{\mathbb{N}}}^{-1}(B_1(0))$, which is impossible because $\phi_n(\chi_{\mathbb{N}}) = 1$ for all $n \in \mathbb{N}$. This shows that $\phi \neq \phi_g$ for all $g \in l^1$.

20. (a) Let $\overline{g} \in (L^p)^*$. There exists $g \in L^q$, where $q := p/(p-1)$, such that $\overline{g}(h) = \int hg$ for all $h \in L^p$. Hence, it suffices to show that $\langle \int f_n g \rangle_{n=1}^\infty$ converges to $\int fg$. To this end, let $\varepsilon \in (0, \infty)$. Set $M := \sup_{n \in \mathbb{N}} \|f_n\|_p + 1$. Since $E \mapsto \int_E |g|^q$ is a finite measure on $\mathcal{M}$ which is absolutely continuous with respect to μ, there exists $\delta \in (0, \infty)$ such that $\int_E |g|^q = |\int_E |g|^q| < \frac{1}{2}(\varepsilon/4M)^q$ for all $E \in \mathcal{M}$ with $\mu(E) < \delta$. Now define $F := \{x \in X \mid 0 < |g(x)|^q\}$ and set $F_m := \{x \in X \mid m^{-1} \leq |g(x)|^q\}$ for each $m \in \mathbb{N}$. Then $\langle F_m \rangle_{m=1}^\infty$ is increasing and $\cup_{m=1}^\infty F_m = F$, so

$$\lim_{m\to\infty} \int_{F_m^c} |g|^q = \lim_{m\to\infty} \left(\int |g|^q - \int_{F_m} |g|^q \right) = \int |g|^q - \lim_{m\to\infty} \int_{F_m} |g|^q = \int |g|^q - \int_F |g|^q = 0.$$

Hence, there exists $m \in \mathbb{N}$ such that $\int_{A^c} |g|^q < \frac{1}{2}(\varepsilon/4M)^q$, where $A := F_m$. Note that $\mu(A) < \infty$ because

$$\mu(A)m^{-1} = \int_A m^{-1} \leq \int_A |g|^q \leq \int |g|^q < \infty.$$

By Egoroff's theorem, it follows that there exists $B \in \mathcal{M}$ such that $B \subseteq A$, $\mu(B) < \delta$ and $\langle f_n|_{A\setminus B} \rangle_{n=1}^\infty$ converges uniformly to $f|_{A\setminus B}$. In particular $\int_{(A\setminus B)^c} |g|^q = \int_{A^c} |g|^q + \int_B |g|^q < (\varepsilon/4M)^q$. By Fatou's lemma

$$\int |f|^p \leq \liminf_{n\to\infty} \int |f_n|^p = \liminf_{n\to\infty} \|f_n\|_p^p \leq M^p,$$

so $f \in L^p$ and $\|f\|_p \leq M$. It follows from Hölder's inequality that

$$\int_{(A\setminus B)^c} |(f_n - f)g| \leq \|f_n - f\|_p \left(\int_{(A\setminus B)^c} |g|^q \right)^{1/q} \leq (\|f_n\|_p + \|f\|_p) \frac{\varepsilon}{4M} \leq \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$. Set $L := \mu(A)\|g\|_q^p + 1$ and choose $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon^p/2^p L$ for all $x \in A \setminus B$ and $n \in \mathbb{N}$ with $n \geq N$. If $n \in \mathbb{N}$ and $n \geq N$, then

$$\int_{A\setminus B} |(f_n - f)g| \leq \left(\int_{A\setminus B} |f_n - f| \right)^{1/p} \|g\|_q \leq \left(\mu(A \setminus B) \frac{\varepsilon^p}{2^p L} \right)^{1/p} \|g\|_q < \frac{\varepsilon}{2}$$

and hence

$$\left| \int f_n g - \int fg \right| \leq \int |(f_n - f)g| = \int_{A\setminus B} |(f_n - f)g| + \int_{(A\setminus B)^c} |(f_n - f)g| < \varepsilon.$$

This shows that $\langle \int f_n g \rangle_{n=1}^\infty$ converges to $\int fg$, as required.

(b) For each $n \in \mathbb{N}$ define $f_n := \chi_{[n,n+1]}$. Then $\sup_{n\in\mathbb{N}} \|f_n\|_1 = \sup_{n\in\mathbb{N}} 1 = 1$ and $\langle f_n \rangle_{n=1}^\infty$ converges to 0 pointwise. Define $g := \chi_E$, where $E := \cup_{n=1}^\infty [2n, 2n+1]$. Clearly $|g| \leq 1$ and $f \mapsto \int fg\, dm$ is in $L^1(\mathbb{R}, m)^*$. However $\langle \int f_n g\, dm \rangle_{n=1}^\infty$ does not converge, and hence $\langle f_n \rangle_{n=1}^\infty$ does not converge weakly, because

$$\int f_n g\, dm = \int \chi_{[n,n+1]\cap E}\, dm = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Similarly, for each $n \in \mathbb{N}$ define $x_n : \mathbb{N} \to \mathbb{C}$ by $x_n := \chi_{\{n\}}$. Then $\sup_{n\in\mathbb{N}} \|x_n\|_1 = \sup_{n\in\mathbb{N}} 1 = 1$ and $\langle x_n \rangle_{n=1}^\infty$ converges to 0 pointwise. Set $g := \chi_{2\mathbb{N}}$, so that $|g| \leq 1$ and hence $x \mapsto \sum_{m=1}^\infty x(m)g(m)$ is in $(l^1)^*$. However

$$\sum_{m=1}^\infty x_n(m)g(m) = g(n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even,} \end{cases}$$

so $\langle\sum_{m=1}^{\infty} x_n(m)g(m)\rangle_{n=1}^{\infty}$ does not converge, and hence $\langle x_n\rangle_{n=1}^{\infty}$ does not converge weakly.

Now let $\langle f_n\rangle_{n=1}^{\infty}$ be a sequence in L^{∞} which converges almost everywhere to $f : X \to \mathbb{C}$. Suppose that $\sup_{n\in\mathbb{N}} \|f_n\|_{\infty} < \infty$ and μ is σ-finite. We aim to show that $\langle\int gf_n\rangle_{n=1}^{\infty}$ converges to $\int gf$ for all $g \in L^1$. To this end, let $g \in L^1$ and set $M := \sup_{n\in\mathbb{N}} \|f_n\|_{\infty}$. For each $n \in \mathbb{N}$ set $E_n := \{x \in X \mid f_n(x) > M\}$. Since $\cup_{n=1}^{\infty} E_n$ has measure zero, it is easily shown that $|f| \leq M$ almost everywhere. It follows that $|g(f_n - f)| \leq |g|(|f_n| + |f|) \leq 2|g|M$ almost everywhere for all $n \in \mathbb{N}$, so by the dominated convergence theorem $\langle\int |g(f_n - f)|\rangle_{n=1}^{\infty}$ converges to $\int 0 = 0$. Since

$$0 \leq \left|\int gf_n - \int gf\right| \leq \int |g(f_n - f)|$$

for all $n \in \mathbb{N}$, the squeeze theorem implies that $\langle\int gf_n\rangle_{n=1}^{\infty}$ converges to $\int gf$, as required.

21. Suppose that $(f_n)_{n=1}^{\infty}$ converges weakly to f in $l^p(A)$. If $a \in A$ then $\chi_{\{a\}} \in l^{p/(p-1)}$ defines a functional $\phi_a \in l^p(A)^*$, so $(f_n(a))_{n=1}^{\infty} = (\phi_a(f_n))_{n=1}^{\infty}$ converges to $\phi_a(f) = f(a)$. Therefore $(f_n)_{n=1}^{\infty}$ converges pointwise to f. If $\phi \in l^p(A)^*$ then $\sup_{n\in\mathbb{N}} |\widehat{f}_n(\phi)| = \sup_{n\in\mathbb{N}} |\phi(f_n)| < \infty$, so $\sup_{n\in\mathbb{N}} \|f_n\|_p = \sup_{n\in\mathbb{N}} \|\widehat{f}_n\| < \infty$ by the uniform boundedness principle. The converse is a special case of exercise 20(a).

22. (a) If $m, n \in \mathbb{N}$ and $m \neq n$ then

$$\begin{aligned}
\langle f_m, f_n\rangle &= \int_0^1 \cos(2\pi mx)\cos(2\pi nx)\,dx \\
&= \frac{1}{2}\int_0^1 (\cos(2\pi(m-n)x) + \cos(2\pi(m+n)x))\,dx \\
&= \frac{1}{2}\left(\frac{\sin(2\pi(m-n)x)}{2\pi(m-n)} + \frac{\sin(2\pi(m+n)x)}{2\pi(m+n)}\right)\Bigg|_0^1 \\
&= \frac{1}{2}\left(\frac{\sin(2\pi(m-n))}{2\pi(m-n)} + \frac{\sin(2\pi(m+n))}{2\pi(m+n)}\right) - \frac{1}{2}\left(\frac{\sin(0)}{2\pi(m-n)} + \frac{\sin(0)}{2\pi(m+n)}\right) \\
&= 0.
\end{aligned}$$

Moreover, if $n \in \mathbb{N}$ then

$$\langle f_n, f_n\rangle = \int_0^1 \cos(2\pi nx)^2\,dx = \frac{1}{2}\int_0^1 (\cos(0) + \cos(4\pi nx))\,dx = \frac{1}{2} + \frac{1}{2}\,\frac{\sin(4\pi nx)}{4\pi n}\Bigg|_0^1 = \frac{1}{2}.$$

This shows that $\langle\sqrt{2}f_n\rangle_{n=1}^{\infty}$ is an orthonormal sequence in L^2, so it converges weakly to 0 by exercise 63 of the previous homework. Therefore $\langle f_n\rangle_{n=1}^{\infty}$ converges weakly to 0.

Let $x \in (0, 1/4\pi)$ and $N \in \mathbb{N}$. Choose $m \in \mathbb{N}$ with $Nx - 1/4 \leq m$. Let $n \in \mathbb{N}$ be the smallest number such that $nx - 1/4 > m$ (which exists because $\mathbb{R}$ is Archimedean). Then $n > N \geq 1$ and hence $(n-1)x - 1/4 \leq m$, in which case $0 < nx - m - 1/4 \leq x < 1/4\pi$. Therefore $|2\pi nx - 2\pi m - \pi/2| < 1/2$. By the mean value theorem and the fact that $|\sin| \leq 1$, it follows that $|\cos(2\pi nx) - 1| = |\cos(2\pi nx) - \cos(2\pi m + \pi/2)| < 1/2$ and hence $\cos(2\pi nx) > 1/2$. This implies that $\langle\cos(2\pi nx)\rangle_{n=1}^{\infty}$ does not converge to 0, so $\langle f_n\rangle_{n=1}^{\infty}$ does not converge to 0 on $(0, 1/4\pi)$. Thus, it is not the case that $\langle f_n\rangle_{n=1}^{\infty}$ converges to 0 almost everywhere.

Note that $\{x \in \mathbb{R} \mid 1/\sqrt{2} \leq \cos(2\pi x)\} = \cup_{m\in\mathbb{Z}}[m + 1/8, m + 3/8] = \cup_{m\in\mathbb{Z}}[(8m+1)/8, (8m+3)/8]$ and hence, for each $n \in \mathbb{N}$, $\{x \in X \mid 1/\sqrt{2} \leq \cos(2\pi nx)\} = \cup_{m=0}^{n-1}[(8m+1)/8n, (8m+3)/8n]$. It follows that $m(\{x \in X \mid 1/\sqrt{2} \leq |f_n(x) - 0|\}) \geq m(\{x \in X \mid 1/\sqrt{2} \leq \cos(2\pi nx)\}) = \sum_{m=0}^{n-1} 2/8n = 2/8$ for all $n \in \mathbb{N}$, which shows that $\langle f_n\rangle_{n=1}^{\infty}$ does not converge to 0 in measure.

(b) If $x \in (0,1]$ there exists $N \in \mathbb{N}$ such that $1/N < x$, in which case $f_n(x) = 0$ for all $n \in \mathbb{N}$ with $n \geq N$. This shows that $\langle f_n \rangle_{n=1}^{\infty}$ converges to 0 on $(0,1]$, hence almost everywhere. If $\varepsilon \in (0,\infty)$ then $m(\{x \in X \mid \varepsilon \leq |f_n(x) - 0|\}) \leq m(\{x \in X \mid 0 < |f_n(x)|\}) = m((0,1/n)) = 1/n$ for all $n \in \mathbb{N}$, which shows that $\langle f_n \rangle_{n=1}^{\infty}$ converges to 0 in measure. Given $p \in [1,\infty]$ it is clear that $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence in L^p, and $f \mapsto \int f$ is in $(L^p)^*$ by Hölder's inequality and the fact that $\chi_X \in L^q$, where q is the conjugate exponent to p. Since $\int f_n = 1 \neq 0$ for all $n \in \mathbb{N}$, it follows that $\langle f_n \rangle_{n=1}^{\infty}$ does not converge to 0 weakly in L^p.

27. Define a measurable function $K : (0,\infty)^2 \to \mathbb{C}$ by $K(x,y) := (x+y)^{-1}$. Note that $K(\lambda x, \lambda y) = \lambda^{-1} K(x,y)$ for all $\lambda, x, y \in (0,\infty)$. Moreover,

$$C_p = \int_0^\infty |K(x,1)| x^{-1/p}\, dx = \int_0^\infty \frac{1}{x^{1+1/p} + x^{1/p}}\, dx \leq \int_0^1 \frac{1}{x^{1/p}}\, dx + \int_1^\infty \frac{1}{x^{1+1/p}}\, dx < \infty.$$

It follows by Theorem 6.20 that $\|Tf\|_p \leq C_p \|f\|_p$ for all $f \in L^p$.

30. (a) If $y \in (0,\infty)$ then

$$\int K(xy) x^{-1/q}\, dx = \int K(z)(z/y)^{-1/q} y^{-1}\, dz = y^{1/q-1} \int K(z) z^{1/p-1}\, dz = y^{-1/p}\phi(p^{-1})$$

and hence, using Hölder's inequality and the fact that $K(xy)f(x) = K(xy)^{1/q} x^{-1/q^2} x^{1/q^2} K(xy)^{1/p} f(x)$,

$$\begin{aligned}\left(\int K(xy)f(x)\, dx\right)^p &\leq \left(\int K(xy) x^{-1/q}\, dx\right)^{p/q} \int x^{p/q^2} K(xy) f(x)^p\, dx \\ &= y^{-1/q}\phi(p^{-1})^{p/q} \int x^{p/q^2} K(xy) f(x)^p\, dx.\end{aligned}$$

Since $p + q = pq$ and hence $p - q = pq - 2q$, it follows by Tonelli's theorem that

$$\begin{aligned}\int \left(\int K(xy)f(x)\, dx\right)^p dy &\leq \int y^{-1/q}\phi(p^{-1})^{p/q} \int x^{p/q^2} K(xy) f(x)^p\, dx\, dy \\ &= \phi(p^{-1})^{p/q} \iint y^{-1/q} x^{p/q^2} K(xy) f(x)^p\, dy\, dx \\ &= \phi(p^{-1})^{p/q} \int x^{p/q^2} f(x)^p \int K(xy) y^{-1/q}\, dy\, dx \\ &= \phi(p^{-1})^{p/q} \int x^{p/q^2} f(x)^p x^{-1/p}\phi(p^{-1})\, dx \\ &= \phi(p^{-1})^{(p+q)/q} \int x^{(p^2-q^2)/(pq^2)} f(x)^p\, dx \\ &= \phi(p^{-1})^p \int x^{(p-q)/q} f(x)^p\, dx \\ &= \phi(p^{-1})^p \int x^{p-2} f(x)^p\, dx.\end{aligned}$$

Thus, by Hölder's inequality (and Tonelli again)

$$\iint K(xy)f(x)g(y)\, dx\, dy = \int \left(\int K(xy)f(x)\, dx\right) g(y)\, dy \leq \phi(p^{-1}) \left(\int x^{p-2} f(x)^p\, dx\right)^{1/p} \|g\|_q.$$

(b) From the long calculation in part (a), if $f \in L^2((0,\infty))$ then

$$\int |Tf(y)|^2\, dy = \int \left|\int K(xy)f(x)\, dx\right|^2 dy \leq \int \left(\int K(xy)|f(x)|\, dx\right)^2 dy \leq \phi(\tfrac{1}{2})^2 \int x^0 |f(x)|^2\, dx$$

and hence $Tf \in L^2((0,\infty))$ with $\|Tf\|_2 \le \phi(\frac{1}{2})\|f\|_2$. Therefore $T : L^2((0,\infty)) \to L^2((0,\infty))$ is well-defined, and it is clearly linear. Moreover $\|T\| \le \phi(\frac{1}{2})$.

32. Since $|K|^2 \in L^1(\mu \times \nu)$, Fubini's theorem implies that $|K_x|^2 \in L^1(\nu)$ for almost all $x \in X$. Therefore $K_x \in L^2(\nu)$ for almost all $x \in X$. By Hölder's inequality $\int |K(x,y)f(y)|\, d\nu(y) \le \|K_x\|_2\|f\|_2$, and hence $Tf(x)$ converges absolutely, for almost all $x \in X$. Fubini's theorem implies that $K^y \in L^2(\mu)$ and hence $K^y f(y) \in L^2(\mu)$, for almost all $y \in Y$. Define $k : Y \to [0,\infty)$ by $k(y) := \|K^y\|_2$. Then

$$\|k\|_2^2 = \int |k(y)|^2\, d\nu(y) = \iint |K^y(x)|^2\, d\mu(x)\, d\nu(y) = \iint |K(x,y)|^2\, d\mu(x)\, d\nu(y) = \|K\|_2^2$$

and hence, by Hölder's inequality

$$\int \|K^y f(y)\|_2\, d\nu(y) = \int \|K^y\|_2 |f(y)|\, d\nu(y) = \|kf\|_1 \le \|k\|_2\|f\|_2 = \|K\|_2\|f\|_2.$$

This implies that $y \mapsto \|K^y f(y)\|_2$ is in $L^1(\nu)$, so by Minkowski's inequality for integrals

$$\int |Tf(x)|^2\, d\mu(x) = \left\| \int K(\cdot,y)f(y)\, d\nu(y) \right\|_2^2 \le \left(\int \|K(\cdot,y)f(y)\|_2\, d\nu(y) \right)^2 \le (\|K\|_2\|f\|_2)^2.$$

This shows that $Tf \in L^2(\mu)$ and $\|Tf\|_2 \le \|K\|_2\|f\|_2$.

33. Since $L^q((0,x)) \subseteq L^1((0,x))$ for all $x \in (0,\infty)$, it is clear that T is a well-defined linear map from $L^q((0,\infty))$ to $\mathbb{C}^{(0,\infty)}$. If $f \in L^q((0,\infty))$, the map $x \mapsto \int_0^x f$ is absolutely continuous on $(0,\infty)$, so Tf is continuous. Given $\varepsilon \in (0,\infty)$ there exists $n \in \mathbb{N}$ such that $\int_n^\infty |f|^q < (\frac{\varepsilon}{2})^q$, by continuity of the finite measure $E \mapsto \int_E |f|^q$. Now

$$\begin{aligned} |Tf(x) - T(f\chi_{(0,n)})(x)| &= \left| x^{-1/p} \int_0^\infty f - x^{-1/p} \int_0^\infty f\chi_{(0,n)} \right| \\ &\le x^{-1/p} \int_0^\infty |f - f\chi_{(0,n)}| \\ &= x^{-1/p} \int \chi_{(0,x)} |f\chi_{[n,\infty)}| \\ &\le x^{-1/p} \|\chi_{(0,x)}\|_p \|f\chi_{[n,\infty)}\|_q \\ &< \frac{\varepsilon}{2} \end{aligned}$$

for all $x \in (0,\infty)$, by Hölder's inequality. In particular, if $x \in (0,\infty)$ and $|Tf(x)| \ge \varepsilon$, then

$$\frac{\varepsilon}{2} \le |T(f\chi_{(0,n)})(x)| \le x^{-1/p} \int_0^n |f|$$

and hence $x \le (\frac{2}{\varepsilon}\|f\chi_{(0,n)}\|_1)^p$. This implies that $\{x \in X \mid \varepsilon \le |Tf(x)|\}$ is bounded, thus compact (because $|Tf|$ is continuous), so $Tf \in C_0((0,\infty))$. Moreover, by Hölder's inequality

$$|Tf(x)| = \left| x^{-1/p} \int f\chi_{(0,x)} \right| \le x^{-1/p} \int |f\chi_{(0,x)}| \le x^{-1/p} \|\chi_{(0,x)}\|_p \|f\|_q = \|f\|_q$$

for all $x \in (0,\infty)$. This shows that $\|Tf\|_u \le \|f\|_q$, so T is a bounded linear map from $L^q((0,\infty))$ to $C_0((0,\infty))$.

34. Set $q := p/(p-1)$. Suppose $p > 2$, so that $p = (p-1)q > q$ and hence $q/p < 1$. By Hölder's inequality

$$\int_0^1 |f'| \le \left(\int_0^1 x|f'(x)|^p\, dx \right)^{1/p} \left(\int_0^1 x^{-q/p}\, dx \right)^{1/q} < \infty$$

and hence $f' \in L^1([0,1])$. Define $g : [0,1] \to \mathbb{C}$ by $g(0) := f(1) - \int_0^1 f'$ and $g|_{(0,\infty)} := f$. Then g is differentiable almost everywhere on $[0,1]$, and $g' \in L^1([0,1])$ because $g' = f'$ almost everywhere. Moreover

$$g(\epsilon) - g(0) = f(\epsilon) - f(1) + \int_0^1 f' = -\int_\epsilon^1 f' + \int_0^1 f' = \int_0^\epsilon f'$$

for all $\epsilon \in (0,1)$. This clearly also holds for $\epsilon \in \{0,1\}$, so g is absolutely continuous on $[0,1]$ and hence $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = g(0)$. If $p = 2$, $\epsilon \in (0, \frac{1}{2})$ and $\delta \in (\epsilon, 2\epsilon)$, then Hölder's inequality implies that

$$\int_\epsilon^\delta |f'| \leq \sqrt{\int_0^1 x|f'(x)|^2\,dx \cdot \int_\epsilon^{2\epsilon} x^{-1}\,dx} = \sqrt{\int_0^1 x|f'(x)|^2\,dx \cdot (\log(2\epsilon) - \log(\epsilon))} = \sqrt{\int_0^1 x|f'(x)|^2\,dx \cdot \log(2)}$$

and hence (where $M := \sqrt{\int_0^1 x|f'(x)|^2\,dx \cdot \log(2)}$)

$$0 \leq \frac{|f(\epsilon)|}{|\log(\epsilon)|^{1/2}} = \frac{\left|f(\delta) - \int_\epsilon^\delta f'\right|}{|\log(\epsilon)|^{1/2}} \leq |\log(\epsilon)|^{-1/2}\left(|f(\delta)| + \int_\epsilon^\delta |f'|\right) \leq |\log(\epsilon)|^{-1/2}\,(|f(\delta)| + M)$$

36. Suppose $f \in$ weak L^p and $\mu(\{x \in X \mid 0 \neq f(x)\}) < \infty$. Fix $q \in (0,p)$. Define $E_0 := \{x \in X \mid 0 < |f(x)| \leq 1\}$, and for each $n \in \mathbb{N}$ set $E_n := \{x \in X \mid 2^{n-1} < |f(x)| \leq 2^n\}$. By the monotone convergence theorem

$$\int |f|^q \leq \int \sum_{n=0}^\infty (2^n)^q \chi_{E_n} = \sum_{n=0}^\infty 2^{nq}\mu(E_n) \leq \mu(E_0) + \sum_{n=1}^\infty 2^{nq}\lambda_f(2^{n-1}) \leq \mu(E_0) + \sum_{n=1}^\infty 2^{nq+p-np}[f]_p^p < \infty,$$

because $E_0 \subseteq \{x \in X \mid 0 \neq f(x)\}$ and $\alpha^p\lambda_f(\alpha) \leq [f]_p^p$ for all $\alpha \in (0,\infty)$ (note that $\sum_{n=1}^\infty 2^{nq-np}$ is a convergent geometric series). This shows that $f \in L^q$.

Now suppose that $f \in (\text{weak } L^p) \cap L^\infty$, and fix $q \in (p,\infty)$. Define $E_0 := \{x \in X \mid 1 < |f(x)|\}$ and for each $n \in \mathbb{N}$ set $E_n := \{x \in X \mid 2^{-n} < |f(x)| \leq 2^{1-n}\}$. Then

$$\begin{aligned}\int |f|^q &\leq \int \left(\|f\|_\infty^q \chi_{E_0} + \sum_{n=1}^\infty (2^{1-n})^q \chi_{E_n}\right)\\ &= \|f\|_\infty^q \mu(E_0) + \sum_{n=1}^\infty 2^{q-nq}\mu(E_n)\\ &\leq \|f\|_\infty^q \lambda_f(1) + \sum_{n=1}^\infty 2^{q-nq}\lambda_f(2^{-n})\\ &\leq \|f\|_\infty^q [f]_p^p + \sum_{n=1}^\infty 2^{q-nq+np}[f]_p^p\end{aligned}$$

and hence $f \in L^q$. It is obvious that $f \in L^q$ for the case $q = \infty$.

38. Let $f : X \to \mathbb{C}$ be a measurable function and suppose that $\sum_{n\in\mathbb{Z}} 2^{np}\lambda_f(2^n) < \infty$. For each $n \in \mathbb{Z}$ define $E_n := \{x \in X \mid 2^n < |f(x)| \leq 2^{n+1}\}$. By the monotone convergence theorem

$$\int |f|^p \leq \int \sum_{n\in\mathbb{Z}} (2^{n+1})^p \chi_{E_n} = 2^p \sum_{n\in\mathbb{Z}} 2^{np}\mu(E_n) \leq 2^p \sum_{n\in\mathbb{Z}} 2^{np}\lambda_f(2^n) < \infty$$

and hence $f \in L^p$. Conversely, suppose that $f \in L^p$. By Proposition 6.24

$$\sum_{n\in\mathbb{Z}} 2^{np}\lambda_f(2^n) = \int_0^\infty \sum_{n\in\mathbb{Z}} 2^{np-n+1}\lambda_f(2^n)\chi_{(2^{n-1},2^n]}$$

$$
\begin{aligned}
&\le 2^p \int_0^\infty \sum_{n\in\mathbb{Z}} 2^{(n-1)(p-1)} \lambda_f(\alpha)\chi_{(2^{n-1},2^n]}(\alpha)\, d\alpha \\
&\le 2^p \int_0^\infty \sum_{n\in\mathbb{Z}} \alpha^{p-1} \lambda_f(\alpha)\chi_{(2^{n-1},2^n]}(\alpha)\, d\alpha \\
&= 2^p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\chi_{(0,\infty)}(\alpha)\, d\alpha \\
&= 2^p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha)\, d\alpha \\
&= 2^p p^{-1} \|f\|_p^p \\
&< \infty.
\end{aligned}
$$

39. If $\phi = \sum_{i=1}^n a_i \chi_{E_i}$ is simple with $\lambda_\phi(0) < \infty$ and $\alpha \in (0,\infty)$, then $\lambda_\phi(\alpha) = 0$ if $\alpha \ge \max\{|a_1|,\ldots,|a_n|\}$ and $\lambda_\phi(\alpha) = \lambda_\phi(0)$ if $\alpha < \min\{|a_1|,\ldots,|a_n|\}$. We may assume that $a_i \neq 0$ for all $i \in \{1,\ldots,n\}$, so

$$
\lim_{\alpha\to\infty} \alpha^p \lambda_\phi(\alpha) = \lim_{\alpha\to\infty} 0 = 0 \quad \text{and} \quad \lim_{\alpha\to 0} \alpha^p \lambda_\phi(\alpha) = \lim_{\alpha\to 0} \alpha^p \lambda_\phi(0) = 0.
$$

Given $\varepsilon \in (0,\infty)$, we may choose $\phi \in L^p$ such that $\|f-\phi\|_p < \frac{\varepsilon}{2^{p+1}}$. By Chebyshev's inequality

$$
0 \le \alpha^p \lambda_f(\alpha) \le \alpha^p \lambda_{f-\phi}\left(\frac{\alpha}{2}\right) + \alpha^p \lambda_\phi\left(\frac{\alpha}{2}\right) \le 2^p \|f-\phi\|_p^p + 2^p \left(\frac{\alpha}{2}\right)^p \lambda_\phi\left(\frac{\alpha}{2}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
$$

provided that $\alpha \in (0,\infty)$ is sufficiently small (large). Therefore $\lim_{\alpha\to 0} \alpha^p \lambda_f(\alpha) = 0$ ($\lim_{\alpha\to\infty} \alpha^p \lambda_f(\alpha) = 0$).

43. Clearly

$$
H\chi_{(0,1)}(x) = \sup_{r\in(0,\infty)} \frac{1}{m((x-r,x+r))} \int_{(x-r,x+r)} |\chi_{(0,1)}(y)|\, dy = \sup_{r\in(0,\infty)} \frac{m((x-r,x+r)\cap(0,1))}{2r}.
$$

If $x \in (-\infty, 0]$ then

$$
m((x-r,x+r)\cap(0,1)) = \begin{cases} 0 & r \in (0,-x) \\ x+r & r \in [-x, 1-x] \\ 1 & r \in (1-x,\infty). \end{cases}
$$

Since $r \mapsto \frac{x}{2r} + \frac{1}{2}$ is increasing on $[-x, 1-x]$, it follows that $H\chi_{(0,1)}(x) = \frac{1}{2-2x}$. Given $x \in (0, \frac{1}{2}]$

$$
m((x-r,x+r)\cap(0,1)) = \begin{cases} 2r & r \in (0,x) \\ x+r & r \in [x, 1-x] \\ 1 & r \in (1-x,\infty). \end{cases}
$$

Since $x + r \le 2r$ for all $r \in [x, 1-x]$ and $1 \le 2r$ for all $r \in (1-x,\infty)$, it follows that $H\chi_{(0,1)}(x) = 1$. Similarly

$$
m((x-r,x+r)\cap(0,1)) = \begin{cases} 2r & r \in (0,1-x) \\ 1-x+r & r \in [1-x, x] \\ 1 & r \in (x,\infty) \end{cases}
$$

whenever $x \in (\frac{1}{2}, 1)$, and hence $H\chi_{(0,1)}(x) = 1$. Given $x \in [1, \infty)$

$$m((x-r, x+r) \cap (0,1)) = \begin{cases} 0 & r \in (0, x-1) \\ 1 - x + r & r \in [x-1, x] \\ 1 & r \in (x, \infty). \end{cases}$$

Since $r \mapsto \frac{1-x}{2r} + \frac{1}{2}$ is increasing on $[x-1, x]$, it follows that $H\chi_{(0,1)}(x) = \frac{1}{2x}$. Hence, if $p \in [1, \infty)$

$$\int |H\chi_{(0,1)}|^p = \int_{-\infty}^0 \frac{dx}{(2-2x)^p} + \int_0^1 dx + \int_1^\infty \frac{dx}{(2x)^p},$$

which is finite iff $p \neq 1$. Also $|H\chi_{(0,1)}| \leq 1$ so $H\chi_{(0,1)} \in L^p$ for all $p \in (1, \infty]$, but not for $p = 1$. Moreover $H\chi_{(0,1)} \in$ weak L^1 because $x \mapsto \frac{1}{x}$ is in weak L^1. Continuing from above

$$\int |H\chi_{(0,1)}|^p = \frac{1}{-2(1-p)(2-2x)^{p-1}}\bigg|_{-\infty}^0 + 1 + \frac{1}{2(1-p)(2x)^{p-1}}\bigg|_1^\infty = \frac{1}{2(p-1)2^{p-1}} + 1 - \frac{1}{2(1-p)2^{p-1}}$$

and hence $\|H\chi_{(0,1)}\|_p = 1 + (p-1)^{-1}2^{1-p}$ for all $p \in (1, \infty)$, so $\|H\chi_{(0,1)}\|_p$ behaves like $(p-1)^{-1}$ as $p \to 1$. It is obvious that $\|\chi_{(0,1)}\|_p = 1$ for all $p \in [1, \infty]$.