4. (a) If $f \in C_c(X,[0,\infty))$ and $a \in (0,\infty)$ then $f^{-1}([a,\infty))$ is a closed subset of the support of f, so it is compact. Moreover, if $N \in \mathbb{N}$ is chosen so that $\frac{1}{N} < a$, then $f^{-1}([a,\infty)) = \cap_{n=N}^{\infty} f^{-1}((a-\frac{1}{n},\infty))$ is a G_δ set.

   (b) Let $K \subseteq X$ be a compact G_δ set, and write $K = \cap_{n=1}^{\infty} U_n$ for some collection $\{U_n\}_{n=1}^{\infty}$ of open subsets of X. For each $n \in \mathbb{N}$ it is clear that $K \subseteq U_n$, so by Urysohn's lemma there exists $f_n \in C_c(X,[0,1])$ such that $f_n(K) = \{1\}$ and $f_n \prec U_n$. Define $f := f_1 \cdot \sum_{n=1}^{\infty} 2^{-n} f_n$ and note that $f \in C_c(X,[0,1])$ because the series converges uniformly and $f \leq f_1$. If $x \in K$ then $f(x) = 1 \cdot \sum_{n=1}^{\infty} 2^{-n} = 1$, but if $x \in K^c$ then $x \notin U_n$ for some $n \in \mathbb{N}$, so $f_n(x) = 0$ and hence $f(x) < 1$. Therefore $K = f^{-1}(\{1\})$.

   (c) Let $\mathcal{B}_X^c$ be the σ-algebra on X generated by the compact G_δ sets. Given $f \in C_c(X)$, by part (a) the positive and negative real and imaginary parts of f are $\mathcal{B}_X^c$-measurable, so f itself is $\mathcal{B}_X^c$-measurable. Therefore $\mathcal{B}_X^0 \subseteq \mathcal{B}_X^c$. Similarly, part (b) implies that $\mathcal{B}_X^c \subseteq \mathcal{B}_X^0$, in which case $\mathcal{B}_X^c = \mathcal{B}_X^0$.

5. (a) Let $\mathcal{B}$ be a countable base for the topology of X, and let $K \subseteq X$ be compact. For each $x \in K^c$ and $y \in K$, there exists an open set $U_{xy} \subseteq X$ such that $y \in U_{xy}$ but $x \notin U_{xy}$. Moreover, there exists $V_{xy} \in \mathcal{B}$ such that $y \in V_{xy} \subseteq U_{xy}$. Since K is compact, for each $x \in K^c$ there is a finite set $Y_x \subseteq K$ such that $K \subseteq \cup_{y \in Y_x} V_{xy}$. Clearly $K = \cap_{x \in K^c} \cup_{y \in Y_x} V_{xy}$, since $x \notin V_{xy}$ for all $x \in K^c$ and $y \in Y_x$. Since the set $\{\cup_{U \in \mathcal{A}} U \mid \mathcal{A} \subseteq \mathcal{B} \text{ is finite}\}$ is a countable collection of open sets, this shows that K is a G_δ set.

   (b) Clearly $\mathcal{B}_X^0 \subseteq \mathcal{B}_X$. Let $U \subseteq X$ be open. For each $x \in U$ there exists a compact neighbourhood $N_x \subseteq U$ of x, and x has a basic open neighbourhood $U_x \subseteq N_x$. For each $x \in U$ note that $\overline{U}_x \subseteq N_x$, so $\overline{U}_x$ is a compact G_δ set and hence $\overline{U}_x \in \mathcal{B}_X^0$. Clearly $U = \cup_{x \in U} \overline{U}_x$, which shows that $U \in \mathcal{B}_X^0$ because $\{U_x\}_{x \in U} \subseteq \mathcal{B}$ is countable. Therefore $\mathcal{B}_X^0$ contains all the open subsets of X, which shows that $\mathcal{B}_X^0 = \mathcal{B}_X$.

7. Exercise 1.10 shows that μ_A is a Borel measure. If $K \subseteq X$ is compact then $\mu_A(K) = \mu(K \cap A) \leq \mu(K) < \infty$. Let $E \in \mathcal{B}_X$, and note that μ is outer regular on $E \cap A$. If $\mu_A(E) = \infty$ then μ_A is clearly outer regular on E. Otherwise, given $\varepsilon \in (0,\infty)$ there is an open set $U \subseteq X$ containing $E \cap A$ such that $\mu(U) < \mu(E \cap A) + \frac{\varepsilon}{2}$. By Proposition 7.7 there is an open set $V \subseteq X$, and a closed set $F \subseteq A^c$, such that $A^c \subseteq V$ and $\mu(V \setminus F) < \frac{\varepsilon}{2}$. It follows that $E \subseteq (E \cap A) \cup A^c \subseteq U \cup V$ and

$$\mu_A(U \cup V) \leq \mu(U \cap A) + \mu(V \cap A) \leq \mu(U) + \mu(V \setminus F) < \mu(E \cap A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mu_A(E) + \varepsilon.$$

Therefore μ_A is outer regular on E. If $K \subseteq E \cap A$ is compact then $K \subseteq E$ and $\mu(K) = \mu_A(K)$. By Corollary 7.6, it follows that μ_A is inner regular on E. Indeed,

$$\mu_A(E) = \sup\{\mu(K) \mid K \subseteq E \cap A \text{ is compact}\} \leq \sup\{\mu_A(K) \mid K \subseteq E \text{ is compact}\} \leq \mu_A(E).$$

8. Since ϕ is integrable ν is finite (in particular, finite on compact sets). Let $E \in \mathcal{B}_X$ and for each $n \in \mathbb{N}$ define $E_n := \{x \in E \mid \phi(x) > \frac{1}{n}\}$. Given $\varepsilon \in (0,\infty)$, by Theorem 3.5 there exists $\delta \in (0,\infty)$ such that $\nu(F) < \frac{\varepsilon}{2}$ for all $F \in \mathcal{B}_X$ with $\mu(F) < \delta$. By Proposition 7.5, for each $n \in \mathbb{N}$ there is a compact set $K_n \subseteq E_n$ such that $\mu(E_n) < \mu(K_n) + \delta$, so that $\nu(E_n \setminus K_n) < \frac{\varepsilon}{2}$. For each $N \in \mathbb{N}$ define $C_N := \cup_{n=1}^{N} K_n$, and set $C := \cup_{n=1}^{\infty} K_n$. By continuity of finite measures $\nu(C \setminus C_N) < \frac{\varepsilon}{2}$ for some $N \in \mathbb{N}$. It follows that

$$\nu(E_n \setminus C_N) \leq \nu(E_n \setminus C) + \nu(C \setminus C_N) \leq \nu(E_n \setminus K_n) + \nu(C \setminus C_N) < \frac{\varepsilon}{2} + \nu(C \setminus C_N)$$

for all $n \in \mathbb{N}$, and hence (since $\phi|_E$ vanishes outside $\cup_{n=1}^{\infty} E_n$)

$$\nu(E \setminus C_N) = \lim_{n \to \infty} \nu(E_n \setminus C_N) \leq \frac{\varepsilon}{2} + \nu(C \setminus C_N) < \varepsilon.$$

Since C_N is compact, this shows that ν is inner regular. Hence, there is a compact set $D \subseteq E^c$ such that $\nu(E^c) < \varepsilon + \nu(D)$, in which case $\nu(D^c) = \nu(X) - \nu(D) < \nu(X) - \nu(E^c) + \varepsilon = \nu(E) + \varepsilon$. Thus ν is outer regular.

10. Let μ be a Radon measure and $f \in L^1(\mu)$ be real-valued. Given $\varepsilon \in (0, \infty)$, there exist LSC functions $g_1 \geq f^+$ and $g_2 \geq f^-$ such that $\int g_1 \, d\mu < \int f^+ \, d\mu + \frac{\varepsilon}{4}$ and $\int g_2 \, d\mu < \int f^- \, d\mu + \frac{\varepsilon}{4}$ (by Proposition 7.14). Moreover, there exist non-negative USC functions $h_1 \leq f^+$ and $h_2 \leq f^-$ such that $\int h_1 \, d\mu > \int f^+ \, d\mu - \frac{\varepsilon}{4}$ and $\int h_2 \, d\mu > \int f^- \, d\mu - \frac{\varepsilon}{4}$. It follows that $h_1 - g_2 \leq f \leq g_1 - h_2$ and

$$\begin{aligned}\int ((g_1 - h_2) - (h_1 - g_2)) \, d\mu &= \int g_1 \, d\mu - \int h_2 \, d\mu - \int h_1 \, d\mu + \int g_2 \, d\mu \\ &< \int f^+ + \frac{\varepsilon}{4} - \int f^- \, d\mu + \frac{\varepsilon}{4} - \int f^+ \, d\mu + \frac{\varepsilon}{4} + \int f^- \, d\mu + \frac{\varepsilon}{4} \\ &= \varepsilon.\end{aligned}$$

By definition $-g_2$ and $-h_2$ are USC and LSC respectively, so $h_1 - g_2$ is USC and $g_1 - h_2$ is LSC.

11. Let $\mathcal{A} := \{A \cap U \mid U \subseteq X \text{ is open and } \mu(A \cap U) \leq \alpha\}$. Since $\mu(A) > 0$ it is clear that $A \neq \varnothing$. Given $a \in A$, by outer regularity there exists an open neighbourhood $U \subseteq X$ of a such that $\mu(U) < \alpha$. Therefore $\mathcal{A} \neq \varnothing$. Let $\mathcal{B} \subseteq \mathcal{A}$ be non-empty and totally ordered (by inclusion). For each $B \in \mathcal{B}$ let $U_B \subseteq X$ be an open set such that $B = A \cap U_B$. Clearly $\cup_{B \in \mathcal{B}} B = A \cap (\cup_{B \in \mathcal{B}} U_B)$. If K is a compact subset of this union, then $\{U_B\}_{B \in \mathcal{B}}$ is an open cover of K, so it has a finite subcover, which can be taken to have a single element U_B because $\mathcal{B}$ is totally ordered. This implies that $K \subseteq A \cap U_B$ and hence $\mu(K) \leq \alpha$, so by inner regularity $\mu(\cup_{B \in \mathcal{B}} B) \leq \alpha$. Therefore $\cup_{B \in \mathcal{B}} B \in \mathcal{A}$ is an upper bound for $\mathcal{B}$, so by Zorn's lemma $\mathcal{A}$ has a maximal element E. Suppose that $\mu(E) < \alpha$. There exists $a \in A \setminus E$ because $\mu(E) < \mu(A)$, and by outer regularity there is an open neighbourhood $U \subseteq X$ of a such that $\mu(U) < \alpha - \mu(E)$. This is a contradiction because $E \subset E \cup \{a\} \subseteq E \cup (A \cap U) \in \mathcal{A}$, so $\mu(E) = \alpha$.

16. Since $\mu^+ \perp \mu^-$, there exists $E \in \mathcal{B}_X$ such that $\mu^+(E) = 0$ and $\mu^-(E^c) = 0$. If $f \in C_c(X, [0, \infty))$ then

$$\begin{aligned}I^+(f) &= \sup\{I(g) \mid g \in C_0(X, \mathbb{R}) \text{ and } 0 \leq g \leq f\} \\ &= \sup\{\textstyle\int g \, d\mu^+ - \int g \, d\mu^- \mid g \in C_0(X, \mathbb{R}) \text{ and } 0 \leq g \leq f\} \\ &\leq \sup\{\textstyle\int g \, d\mu^+ \mid g \in C_0(X, \mathbb{R}) \text{ and } 0 \leq g \leq f\} \\ &= \textstyle\int f \, d\mu^+.\end{aligned}$$

Given $\varepsilon \in (0, \infty)$, by Lusin's theorem there exists $\phi \in C_c(X)$ such that $\phi = f\chi_{E^c}$ except on a set $F \in \mathcal{B}_X$ with $|\mu|(F) < \varepsilon$. Clearly $\psi := \min\{f, \operatorname{Re}(\phi)^+\}$ has the same properties, but also $0 \leq \psi \leq f$. It follows that

$$\begin{aligned}I^+(f) &\geq I(\psi) \\ &= \textstyle\int \psi \, d\mu^+ - \int \psi \, d\mu^- \\ &= \textstyle\int_F \psi \, d\mu^+ + \int_{F^c} \psi \, d\mu^+ - \int_F \psi \, d\mu^- - \int_{F^c} \psi \, d\mu^- \\ &\geq \textstyle\int_{F^c} f\chi_{E^c} \, d\mu^+ - \int_F \|f\|_u \, d\mu^+ - \int_{F^c} f\chi_{E^c} \, d\mu^- \\ &= \textstyle\int_{F^c} f(\chi_{E^c} + \chi_E) \, d\mu^+ - \|f\|_u \mu^+(F) \\ &\geq \textstyle\int_{F^c} f \, d\mu^+ + \int_F f \, d\mu^+ - \int_F f \, d\mu^+ - \varepsilon \|f\|_u \\ &\geq \textstyle\int f \, d\mu^+ - 2\varepsilon \|f\|_u.\end{aligned}$$

Since $\|f\|_u$ is fixed, it follows that $I^+(f) \geq \int f\,d\mu^+$ and hence $I^+(f) = \int f\,d\mu^+$. This holds for all $f \in C_c(X)$ by linearity. The Riesz representation theorem then implies that μ^+ is the unique Radon measure associated to I^+. It also implies that μ^- is the unique Radon measure associated to I^-, because

$$I^-(f) = I^+(f) - I(f) = \int f\,d\mu^+ - \int f\,d\mu = \int f\,d\mu^+ - \int f\,d\mu^+ + \int f\,d\mu^- = \int f\,d\mu^-$$

for all $f \in C_c(X)$.

22. Suppose that $\langle f_n\rangle_{n=1}^\infty$ converges weakly to $f \in C_0(X)$. Then $\{\widehat{f_n}(I)\}_{n=1}^\infty = \{I(f_n)\}_{n=1}^\infty$ is bounded for all $I \in C_0(X)^*$, so by the principle of uniform boundedness $\{\|f_n\|_u\}_{n=1}^\infty = \{\|\widehat{f_n}\|\}_{n=1}^\infty$ is also bounded. Let $x \in X$ and define $I_x : C_0(X) \to \mathbb{C}$ by $I_x(g) := g(x)$. Clearly I_x is linear, and $|I_x(g)| = |g(x)| \leq \|g\|_u$ for all $g \in C_0(X)$, so $I_x \in C_0(X)$, implying that $\langle f_n(x)\rangle_{n=1}^\infty = \langle I_x(f_n)\rangle_{n=1}^\infty$ converges to $I_x(f) = f(x)$. In other words $\langle f_n\rangle_{n=1}^\infty$ converges pointwise to f.

Conversely, suppose that $M := \sup\{\|f_n\|_u\}_{n=1}^\infty < \infty$ and $\langle f_n\rangle_{n=1}^\infty$ converges pointwise to $f \in C_0(X)$. Given $\mu \in M(X)$, we may decompose μ into four finite positive measures, so that the constant function M is integrable with respect to each of them. By the dominated convergence theorem $\langle \int f_n\,d\nu\rangle_{n=1}^\infty$ converges to $\int f\,d\nu$ for each of these measures ν, implying that $\langle I_\mu(f_n)\rangle_{n=1}^\infty = \langle \int f_n\,d\mu\rangle_{n=1}^\infty$ converges to $\int f\,d\mu = I_\mu(f)$. Therefore $\langle f_n\rangle_{n=1}^\infty$ converges weakly to f.