3. (a) Note that $\eta^{(0)}(t) = 1 \cdot e^{-1/t}$ for all $t \in (0, \infty)$, where 1 is a polynomial of degree 0. Given $k \in \mathbb{N}$, suppose that $\eta^{(k-1)}(t) = P_{k-1}(1/t)e^{-1/t}$ for all $t \in (0, \infty)$, where $P_{k-1}(x)$ is some polynomial of degree $2(k-1)$. By the product rule and the chain rule, $\eta^{(k)}(t) = P'_{k-1}(1/t)(-t^{-2})e^{-1/t} + P_{k-1}(1/t)e^{-1/t}t^{-2} = P_k(1/t)e^{-1/t}$ for all $t \in (0, \infty)$, where $P_k(x) := x^2(P_{k-1}(x) - P'_{k-1}(x))$ is a polynomial of degree $2 + 2(k-1) = 2k$. It follows by induction that this result holds for all $k \in \mathbb{N} \cup \{0\}$.

   (b) By definition $\eta^{(0)}(0) = 0$. Given $k \in \mathbb{N}$, suppose that $\eta^{(k-1)}(0) = 0$. Note that

$$\lim_{n\to\infty} \left| \frac{(n+1)e^{-(n+1)}}{ne^{-n}} \right| = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{e} = \frac{1}{e} < 1,$$

so that $\sum_{n=1}^{\infty} ne^{-n}$ converges by the ratio test. This implies that $\lim_{n\to\infty} ne^{-n} = 0$, and hence

$$\lim_{t\searrow 0} \frac{\eta^{(k-1)}(t) - \eta^{(k-1)}(0)}{t} = \lim_{t\searrow 0} \frac{P_{k-1}(t)e^{-1/t}}{t} = P_{k-1}(0) \lim_{t\searrow 0} \frac{e^{-1/t}}{t} = 0.$$

The left-hand derivative of $\eta^{(k-1)}$ at 0 is clearly 0, so $\eta^{(k)}(0) = 0$. By induction, $\eta^{(k)}(0) = 0$ for all $k \in \mathbb{N}$.

4. This question is surprisingly subtle, and one of the few instances where it is important to keep track of null sets explicitly. It can be shown that $A_r f$ is uniformly continuous for arbitrary $f \in L^\infty$ and $r \in (0, \infty)$, but the δ constructed depends on r, which will cause issues later. Instead, let $\varepsilon \in (0, \infty)$ and take $\delta \in (0, \infty)$ such that $\|\tau_y f - f\|_\infty < \frac{\varepsilon}{2}$ for all $y \in B_\delta(0)$. If $r \in (0, \infty)$ and $x, y \in \mathbb{R}^n$ satisfy $|x - y| < \delta$, then

$$\begin{aligned}
|A_r f(x) - A_r f(y)| &= \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(z)\,dz - \frac{1}{m(B_r(y))} \int_{B_r(y)} f(z)\,dz \right| \\
&= \left| \frac{1}{m(B_r(y))} \int_{B_r(y)} f(z + x - y)\,dz - \frac{1}{m(B_r(y))} \int_{B_r(y)} f(z)\,dz \right| \\
&\le \frac{1}{m(B_r(y))} \int_{B_r(y)} |\tau_{y-x} f(z) - f(z)|\,dz \\
&\le \|\tau_{y-x} f - f\|_\infty \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

In particular, $A_r f$ is uniformly continuous. If $x \in \mathbb{R}^n$ and $r, s \in (0, \delta)$, then the above inequality gives

$$\begin{aligned}
|A_r f(x) - A_s f(x)| &= \left| A_r f(x) - \frac{1}{m(B_r(x))} \int_{B_r(x)} A_s f(y)\,dy + \frac{1}{m(B_r(x))} \int_{B_r(x)} A_s f(y)\,dy - A_s f(x) \right| \\
&\le \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} (f(y) - A_s f(y))\,dy \right| + \frac{1}{m(B_r(x))} \int_{B_r(x)} |A_s f(y) - A_s f(x)|\,dy \\
&< \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} \left(f(y) - \frac{1}{m(B_s(y))} \int_{B_s(y)} f(z)\,dz \right) dy \right| + \frac{1}{m(B_r(x))} \int_{B_r(x)} \frac{\varepsilon}{2}\,dy \\
&\le \frac{1}{m(B_r(x))m(B_s(y))} \int_{B_r(x)} \int_{B_s(y)} |f(y) - f(z)|\,dz\,dy + \frac{\varepsilon}{2} \\
&= \frac{1}{m(B_r(x))m(B_s(y))} \int_{B_r(x)} \int_{B_s(0)} |f(y) - \tau_{-z} f(y)|\,dz\,dy + \frac{\varepsilon}{2} \\
&= \frac{1}{m(B_r(x))m(B_s(y))} \int_{B_s(0)} \int_{B_r(x)} |f(y) - \tau_{-z} f(y)|\,dy\,dz + \frac{\varepsilon}{2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{m(B_r(x))m(B_s(y))}\int_{B_s(0)}\int_{B_r(x)}\|\tau_{-z}f-f\|_\infty\,dy\,dz+\frac{\varepsilon}{2}\\
&<\varepsilon.
\end{aligned}
$$

Therefore $(A_{1/n}f)_{n=1}^\infty$ is uniformly Cauchy, so it converges uniformly to a function g which is uniformly continuous (by a standard argument). Theorem 3.18 implies that $f=g$ almost everywhere.

7. Choose $R\in(0,\infty)$ so that g is supported on $B_R(0)$. The partial derivatives of g (up to order k) are also supported on $B_R(0)$, and hence they are bounded. If $x\in\mathbb{R}^n$ and $y\in B_1(x)$ then $f*g(y)=\int f(z)g(y-z)\,dz=(f\chi_{B_{R+1}(x)})*g(y)$, so $(f*g)|_{B_1(x)}\in C^k(B_1(x))$ by Proposition 8.10. It follows that $f*g\in C^k(\mathbb{R}^n)$.

8. If $x\in\mathbb{R}^n$, e_j is the j^{th} standard basis vector in $\mathbb{R}^n$ and $t\in\mathbb{R}$ is non-zero then

$$
\begin{aligned}
0&\leq\left|\frac{(f*g)(x+te_j)-(f*g)(x)}{t}-(\partial_jf)*g\right|\\
&=\left|t^{-1}\int(f(x+te_j-y)-f(x-y))g(y)\,dy-\int(\partial_jf)(x-y)g(y)\,dy\right|\\
&=\left|\int(t^{-1}((\tau_{-te_j}f)(x-y)-f(x-y))-(\partial_jf)(x-y))g(y)\,dy\right|\\
&\leq\int|t^{-1}((\tau_{-te_j}f)(x-y)-f(x-y))-(\partial_jf)(x-y)||g(y)|\,dy\\
&\leq\left(\int|t^{-1}((\tau_{-te_j}f)(x-y)-f(x-y))-(\partial_jf)(x-y)|^p\,dy\right)^{1/p}\|g\|_q\\
&=\left(\int|t^{-1}((\tau_{-te_j}f)(y)-f(y))-(\partial_jf)(y)|^p\,dy\right)^{1/p}\|g\|_q\\
&=\|t^{-1}(\tau_{-te_j}f-f)-\partial_jf\|_p\|g\|_q
\end{aligned}
$$

by Hölder's inequality, so by the squeeze theorem $\partial_j(f*g)=(\partial_jf)*g$.

9. Suppose that f is absolutely continuous on every bounded interval and that $f'\in L^p$. If $t,x\in\mathbb{R}$ and $t\neq0$ then

$$
\begin{aligned}
(t^{-1}(\tau_{-t}f-f)-f')(x)&=t^{-1}((\tau_{-t}f)(x)-f(x))-f'(x)\\
&=t^{-1}(f(x+t)-f(x))-f'(x)\\
&=t^{-1}\int_x^{x+t}f'(y)\,dy-f'(x)\\
&=\int_0^t t^{-1}(f'(x+y)-f'(x))\,dy\\
&=\int_{[t,0]\cup[0,t]}|t|^{-1}(f'(x+y)-f'(x))\,dy
\end{aligned}
$$

and hence, by Minkowski's inequality for integrals

$$
\begin{aligned}
\|t^{-1}(\tau_{-t}f-f)-f'\|_p&\leq\int_{[t,0]\cup[0,t]}\left(\int|t^{-1}(f'(x+y)-f'(x))|^p\,dx\right)^{1/p}dy\\
&=|t|^{-1}\int_{[t,0]\cup[0,t]}\left(\int|(\tau_{-y}f')(x)-f'(x)|^p\,dx\right)^{1/p}dy
\end{aligned}
$$

$$= |t|^{-1} \int_{[t,0]\cup[0,t]} \|\tau_{-y} f' - f'\|_p \, dy.$$

Given $\varepsilon \in (0, \infty)$, since $f' \in L^p$ there exists $\delta \in (0, \infty)$ such that $\|\tau_{-y} f' - f'\|_p < \varepsilon$ for all $y \in (-\delta, \delta)$. In particular, if $|t| < \delta$ then $\int_{[t,0]\cup[0,t]} \|\tau_{-y} f' - f'\|_p \, dy \le \int_{[t,0]\cup[0,t]} \varepsilon \, dy = \varepsilon |t|$, in which case $\|t^{-1}(\tau_{-t} f - f) - f'\|_p \le \varepsilon$. This shows that f' is the L^p derivative of f.

Conversely, suppose that f has an L^p derivative h. Define $\phi \in C_c(X)$ by $\phi(x) := (1 - |x|)\chi_{[-1,1]}(x)$. Clearly $0 \le \phi \le 4(1 + |x|)^{-2}$ and $\int \phi = 1$, so $\langle f * \phi_{1/n} \rangle_{n=1}^{\infty}$ converges to f pointwise on L_f. If $a, b \in L_f$ and $a < b$ then $h \in L^p([a, b]) \subseteq L^1([a, b])$, and the previous exercise implies that $\langle (f * \phi_{1/n})' \rangle_{n=1}^{\infty} = \langle h * \phi_{1/n} \rangle_{n=1}^{\infty}$. This sequence converges to h in $L^1([a, b])$. Moreover $f * \phi_{1/n}$ is absolutely continuous for all $n \in \mathbb{N}$, because

$$|(f * \phi_{1/n})'(x)| = |(h * \phi_{1/n})(x)| \le \int |h(x - y)\phi_{1/n}(y)| \, dy = \int_{-1/n}^{1/n} |h(x - y)| n\phi(ny) \, dy \le n\|h\|_{L^1[a-1/n, b+1/n]}$$

for all $x \in [a, b]$ (so its derivative is bounded on $[a, b]$). It follows that

$$f(x) - f(a) = \lim_{n\to\infty} ((f * \phi_{1/n})(x) - (f * \phi_{1/n})(a)) = \lim_{n\to\infty} \int_a^x h * \phi_{1/n} = \int_a^x h$$

for all $x \in L_f \cap [a, b]$. By redefining f on a null set, it follows that $f \in AC([a, b])$ and $f' = h$ almost everywhere on $[a, b]$. Given an interval with endpoints not in L_f, there is a larger interval whose endpoints do lie in L_f, so f is absolutely continuous, with $f' = h$, on every bounded interval. This implies that $f' = h$ almost everywhere, in which case $f' \in L^p$.

10. Let $x \in \mathbb{R}^n$ and $t \in (0, \infty)$. Set $E_0 := B_t(0)$ and for each $k \in \mathbb{N}$ define $E_k := B_{2^k t}(0) \setminus B_{2^{k-1} t}(0)$. Then

$$\begin{aligned}
\int_{E_0} |f(x - y)\phi_t(y)| \, dy &= \int_{E_0} |f(x - y)| t^{-n} |\phi(t^{-1} y)| \, dy \\
&\le \frac{1}{t^n} \int_{E_0} |f(x - y)| C(1 + |t^{-1} y|)^{-n-\varepsilon} \, dy \\
&\le \frac{C \cdot m(B_1(0))}{m(B_t(0))} \int_{E_0} |f(x - y)| \, dy \\
&= \frac{C \cdot m(B_1(0))}{m(B_t(x))} \int_{B_t(x)} |f(y)| \, dy \\
&\le C \cdot m(B_1(0)) \cdot Hf(x).
\end{aligned}$$

Moreover, if $k \in \mathbb{N}$ then

$$\begin{aligned}
\int_{E_k} |f(x - y)\phi_t(y)| \, dy &= \int_{E_k} |f(x - y)| t^{-n} |\phi(t^{-1} y)| \, dy \\
&\le \frac{1}{t^n} \int_{E_k} |f(x - y)| C(1 + |t^{-1} y|)^{-n-\varepsilon} \, dy \\
&\le \frac{C}{t^n} \int_{E_k} |f(x - y)| (1 + 2^{k-1})^{-n-\varepsilon} \, dy \\
&\le \frac{C \cdot 2^{kn} m(B_1(0))}{2^{(k-1)(n+\varepsilon)} \cdot m(B_{2^k t}(0))} \int_{B_{2^k t}(0)} |f(x - y)| \, dy \\
&= \frac{C \cdot m(B_1(0))}{2^{(k-1)\varepsilon - n} \cdot m(B_{2^k t}(x))} \int_{B_{2^k t}(x)} |f(y)| \, dy
\end{aligned}$$

$$\leq \frac{C \cdot m(B_1(0))}{2^{(k-1)\varepsilon - n}} Hf(x).$$

It follows that

$$|f * \phi_t(x)| \leq \int |f(x-y)\phi_t(y)|\,dy \leq C \cdot m(B_1(0)) \left(1 + \sum_{k=1}^{\infty} 2^{n-(k-1)\varepsilon}\right) Hf(x) = C \cdot m(B_1(0)) \left(1 + \frac{2^n}{1-2^{-\varepsilon}}\right) Hf(x)$$

and hence $M_\phi f \leq C \cdot m(B_1(0))(1 + \frac{2^n}{1-2^{-\varepsilon}})Hf$.

13. (a) Clearly $\widehat{f}(0) = \int_0^1 (\frac{1}{2} - x)\,dx = 0$. If $\kappa \in \mathbb{Z} \setminus \{0\}$ then we integrate by parts:

$$\widehat{f}(\kappa) = \int_0^1 \left(\frac{1}{2} - x\right) e^{-2\pi i\kappa x}\,dx = \left(\frac{1}{2} - x\right) \frac{e^{-2\pi i\kappa x}}{-2\pi i\kappa}\bigg|_0^1 + \int_0^1 \frac{e^{-2\pi i\kappa x}}{-2\pi i\kappa}\,dx = \frac{1+1}{4\pi i\kappa} + \frac{e^{-2\pi i\kappa x}}{-4\pi^2\kappa^2}\bigg|_0^1 = \frac{1}{2\pi i\kappa}.$$

(b) By Plancherel's theorem

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 4\pi^2 \sum_{k=1}^{\infty} |\widehat{f}(k)|^2 = 2\pi^2 \sum_{\kappa\in\mathbb{Z}} |\widehat{f}(\kappa)|^2 = 2\pi^2 \|f\|^2 = 2\pi^2 \int_0^1 \left(\frac{1}{2} - x\right)^2 dx = -\frac{2\pi^2}{3}\left(\frac{1}{2} - x\right)^3\bigg|_0^1 = \frac{\pi^2}{6}.$$

15. (a) Let $a \in (0, \infty)$, and note that $\chi_{[-a,a]} \in L^1$. If $\xi \in \mathbb{R} \setminus \{0\}$ then

$$\widehat{\chi}_{[-a,a]}(\xi) = \int e^{-2\pi i\xi x} \chi_{[-a,a]}(x)\,dx = \frac{e^{-2\pi i\xi x}}{-2\pi i\xi}\bigg|_{x=-a}^{x=a} = \frac{e^{2\pi i\xi a} - e^{-2\pi i\xi a}}{2\pi i\xi} = \frac{2ia\sin(2\pi\xi a)}{2\pi i\xi a} = 2a\,\mathrm{sinc}(2\xi a),$$

and it is clear that $\widehat{\chi}_{[-a,a]}(0) = \int \chi_{[-a,a]} = 2a = 2a\,\mathrm{sinc}(2a \cdot 0)$. It follows that $\chi^\vee_{[-a,a]}(\xi) = \widehat{\chi}_{[-a,a]}(-\xi) = 2a\,\mathrm{sinc}(-2\xi a) = 2a\,\mathrm{sinc}(2\xi a)$ for all $\xi \in \mathbb{R}$.

(b) Since the Fourier transform is linear, it is clear that $\mathcal{H}_a$ is a subspace of L^2. If $\langle f_n\rangle_{n=1}^\infty$ is a sequence in $\mathcal{H}_a$ which converges to $f \in L^2$, then $\langle \widehat{f_n}\rangle_{n=1}^\infty$ converges to $\widehat{f}$ in L^2 (because the Fourier transform is unitary), and some subsequence of $\langle \widehat{f_n}\rangle_{n=1}^\infty$ converges to $\widehat{f}$ pointwise almost everywhere (because it converges in measure). In particular $\widehat{f}(\xi) = 0$ for almost all $\xi \in \mathbb{R} \setminus [-a, a]$, which implies that $f \in \mathcal{H}_a$. This shows that $\mathcal{H}_a$ is a closed subspace of L^2, so it is a Hilbert space.

Define $\phi := (2a)^{-1/2}\chi_{[-a,a]}$ and $f := \widehat{\phi}$. If $\psi \in \mathcal{S}$ then $\mathcal{F}^2\psi$ is the reflection of $(\widehat{\psi})^\vee = \psi$ about 0. Therefore $\mathcal{F}^2$ agrees with the reflection operator on a dense subset of L^2, so the two operators are equal and $\widehat{f}$ is the reflection of ϕ about 0. This shows that $\widehat{f} = \phi$, so $f \in \mathcal{H}_a$. For each $k \in \mathbb{Z}$ define $f_k := \tau_{k/2a} f$, so that $\widehat{f_k}(\xi) = e^{-\pi i\xi k/a}\widehat{f}(\xi)$ for all $\xi \in \mathbb{R}$, and in particular $f_k \in \mathcal{H}_a$. If $j, k \in \mathbb{Z}$ then

$$\langle f_j, f_k\rangle = \langle \widehat{f_j}, \widehat{f_k}\rangle = \int e^{-\pi i\xi j/a}\widehat{f}(\xi)e^{\pi i\xi k/a}\overline{\widehat{f}(\xi)}\,d\xi = \int e^{\pi i\xi(k-j)/a}|\phi(\xi)|^2\,d\xi = \frac{1}{2a}\int_{-a}^{a} e^{\pi i\xi(k-j)/a}\,d\xi,$$

and if $j \neq k$ it follows that $\langle f_j, f_k\rangle = \frac{1}{2\pi i(k-j)}(e^{\pi i(k-j)} - e^{\pi i(j-k)}) = 0$. Moreover, $\langle f_k, f_k\rangle = \frac{1}{2a}\int_{-a}^{a} 1 = 1$. This shows that $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal set in $\mathcal{H}_a$. If $g \in \mathcal{H}_a$ and $g \perp f_k$ for all $k \in \mathbb{Z}$, then

$$\int_{-a}^{a} \widehat{g}(\xi)e^{\pi i\xi k/a}\,d\xi = \sqrt{2a}\int \widehat{g}(\xi)e^{\pi i\xi k/a}\phi(\xi)\,d\xi = \sqrt{2a}\langle \widehat{g}, \widehat{f_k}\rangle = \sqrt{2a}\langle g, f_k\rangle = 0$$

for all $k \in \mathbb{Z}$. This implies that $\widehat{g}|_{[-a,a]} \in \mathcal{M}^\perp$, where $\mathcal{M}$ is the closed span of the collection of functions of the form $\xi \mapsto e^{-\pi i\xi k/a}$ (where $k \in \mathbb{Z}$). But $\mathcal{M} = L^2([-a, a])$ by the Stone-Weierstraß theorem and the fact that the inclusion $C([-a, a]) \hookrightarrow L^2([-a, a])$ is a bounded linear map with dense range (this is essentially Theorem 8.20). Therefore $\widehat{g}|_{[-a,a]} = 0$ almost everywhere, so $\widehat{g} = 0$ almost everywhere and hence $g = 0$ almost everywhere. This shows that $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}_a$.

(c) Given $f \in \mathcal{H}_a$, the series $\sum_{k\in\mathbb{Z}}\langle f, f_k\rangle f_k = \sum_{k\in\mathbb{Z}}\langle \widehat{f}, \widehat{f_k}\rangle f_k$ converges to f in $\mathcal{H}^a$ (thus in L^2). If $k \in \mathbb{Z}$ then

$$\langle \widehat{f}, \widehat{f_k}\rangle = \int \widehat{f}(x)e^{\pi ixk/a}\overline{\phi(x)}\,dx = \int_{-a}^{a} \frac{e^{\pi ixk/a}}{\sqrt{2a}}\widehat{f}(x)\,dx = \frac{1}{\sqrt{2a}}\int e^{\pi ixk/a}\widehat{f}(x)\,dx = \frac{\mathcal{F}^2 f(-k/2a)}{\sqrt{2a}} = \frac{f(k/2a)}{\sqrt{2a}}.$$

Let $Z \subseteq \mathbb{Z}$. By Hölder's inequality

$$\sum_{k\in Z}|\langle f, f_k\rangle f_k(\xi)| \le \sqrt{\sum_{k\in Z}|\langle f, f_k\rangle|^2\sum_{k\in Z}|f_k(\xi)|^2} \le \sqrt{\sum_{k\in Z}|\langle f, f_k\rangle|^2\sum_{k\in\mathbb{Z}}2a|\operatorname{sinc}(2a\xi-k)|^2}$$

for all $\xi \in \mathbb{R}$. Parseval's identity shows that the first sum $\sum_{k\in Z}|\langle f, f_k\rangle|^2$ can be made small by choosing Z appropriately, so the series $\sum_{k\in Z}\langle f, f_k\rangle f_k$ will be uniformly Cauchy provided that the second sum is uniformly bounded. It suffices to show this for $\xi \in [0, \frac{1}{2a})$ because

$$\sum_{k\in\mathbb{Z}}2a|\operatorname{sinc}(2a(\xi+i/2a)-k)|^2 = \sum_{k\in\mathbb{Z}}2a|\operatorname{sinc}(2a\xi+i-k)|^2 = \sum_{j\in\mathbb{Z}}2a|\operatorname{sinc}(2a\xi-j)|^2$$

for all $\xi \in \mathbb{R}$ and $i \in \mathbb{Z}$. Given $\xi \in [0, \frac{1}{2a})$ define $Z := \mathbb{Z}\setminus\{-1,0,1\}$, so that (since $|\operatorname{sinc}| \le 1$)

$$\begin{aligned}\sum_{k\in\mathbb{Z}}2a|\operatorname{sinc}(2a\xi-k)|^2 &\le 6a + \sum_{k\in Z}2a\frac{|\sin(2\pi a\xi)|^2}{|2\pi a\xi-\pi k|^2}\\ &\le 6a + \sum_{k\in Z}\frac{2a}{|2\pi a\xi-\pi k|^2}\\ &\le 6a + \sum_{k\in Z}\frac{2a}{(\pi|k|-2\pi a\xi)^2}\\ &\le 6a + \sum_{k\in Z}\frac{2a}{(\pi|k|-\pi)^2}\\ &\le 6a + \sum_{k\in Z}\frac{2a}{(\pi|k|/2)^2}\\ &= 6a + \sum_{k\in Z}\frac{8a}{\pi^2|k|^2}\\ &= 6a + \frac{16a}{\pi^2}\sum_{k=2}^{\infty}\frac{1}{k^2}\\ &< \infty.\end{aligned}$$

This shows that $\sum_{k\in\mathbb{Z}} f(k/2a)\operatorname{sinc}(2a\xi-k)$ is uniformly Cauchy for all $\xi \in \mathbb{R}$. It is clear that sinc is continuous, and in fact $\operatorname{sinc} \in C_0$ because $|\operatorname{sinc}(\xi)| \le |\pi\xi|^{-1}$ for all $\xi \in \mathbb{R}\setminus\{0\}$. Hence, the above series converges uniformly to some $g \in C_0$. A subsequence of its partial sums converges pointwise almost everywhere to f (because the series converges to f in measure). This implies that $f = g$ almost everywhere.

16. (a) If $k \in \mathbb{N}$ and $x \in \mathbb{R}$ then

$$f_k(x) = \int \chi_{[-1,1]}(x-y)\chi_{[-k,k]}(y)\,dy = \int \chi_{[x-1,x+1]\cap[-k,k]} = \begin{cases}0, & \text{if } x \in (-\infty,-k-1]\cup[k+1,\infty)\\ 2, & \text{if } x \in [1-k,k-1]\\ k+1-x & \text{if } x \in [k-1,k+1]\\ k+1+x & \text{if } x \in [-k-1,1-k].\end{cases}$$

It clearly follows that $\|f_k\|_u = 2$.

(b) If $k \in \mathbb{N}$ and $\xi \in \mathbb{R}$ then

$$f_k^\vee(\xi) = \widehat{f}_k(-\xi) = \widehat{\chi}_{[-1,1]}(-\xi)\widehat{\chi}_{[-k,k]}(-\xi) = 2\,\mathrm{sinc}(-2\xi) \cdot 2k\,\mathrm{sinc}(-2\xi k) = \frac{4k}{(\pi\xi)^2}\sin(2\pi\xi)\sin(2\pi k\xi).$$

Therefore

$$\|f_k^\vee\|_1 = \int \frac{4k|\sin(2\pi\xi)\sin(2\pi k\xi)|}{(\pi\xi)^2}\,d\xi = \int \frac{4|\sin(\pi\zeta/6k)\sin(\pi\zeta/6)|}{12(\pi\zeta/12k)^2}\,d\zeta = \int \frac{48k^2|\sin(\pi\zeta/6k)\sin(\pi\zeta/6)|}{\pi^2\zeta^2}\,d\zeta.$$

Note that $|\sin(\pi\zeta/6)| \geq \frac{1}{2}$ for all $\zeta \in \cup_{i\in\mathbb{Z}}([1,5]+6i)$ and (provided that $k \geq 6$) at least $m := \lceil \frac{k}{2} \rceil$ of these intervals are contained in $[k, 5k]$. Indeed, at least one of them begins in $[k, k+5]$, after which there is an available length of $4k-5 \geq 6m-2$, which is enough room for m such intervals. Hence, there exists $a \in 6\mathbb{Z}$ such that $\cup_{i=0}^{m-1}[a_i, b_i] \subseteq [k, 5k]$, where $a_i := 1 + a + 6i$ and $b_i := a_i + 4$ for all $i \in \{0, \dots, m-1\}$. Therefore

$$\|f_k^\vee\|_1 \geq \sum_{i=0}^{m-1}\int_{a_i}^{b_i} \frac{12k^2}{\pi^2\zeta^2}\,d\zeta = \sum_{i=0}^{m-1}\frac{12k^2}{\pi^2}\left(\frac{1}{a_i} - \frac{1}{b_i}\right) \geq \sum_{i=0}^{m-1} k^2\frac{(b_i - a_i)}{a_i b_i} \geq \sum_{i=0}^{m-1}\frac{4k^2}{(5k)^2} = \frac{4m}{25} \geq \frac{2k}{25},$$

which implies that $\lim_{k\to\infty}\|f_k^\vee\|_1 = \infty$.

(c) If $f \in L^1$ and $\widehat{f} = 0$ then $\widehat{f} \in L^1$ and hence $f = (\widehat{f})^\vee = 0$ almost everywhere. In particular $\mathcal{F}|_{L^1}$ is injective (it has trivial kernel). Moreover $\mathcal{F}|_{L^1} \in L(L^1, C_0)$ because $|\widehat{f}(\xi)| = |\int e^{-2\pi i\xi x} f(x)\,dx| \leq \|f\|_1$ for all $f \in L^1$ and $\xi \in \mathbb{R}$, by definition. If $\mathcal{F}(L^1) = C_0$, then $\mathcal{F}|_{L^1}$ has an inverse $\mathcal{G} \in L(C_0, L^1)$ by the open mapping theorem (since L^1 and C_0 are Banach spaces). This contradicts the previous parts, because $\|\mathcal{G}\| \geq \frac{\|\mathcal{G}f_k\|_1}{\|f_k\|_u} \geq \frac{k}{25}$ for all $k \in \mathbb{N}$ with $k \geq 6$.

17. (a) Note that

$$\int |f(x)|\,dx = \int_0^\infty e^{-2\pi x}x^{a-1}\,dx = \frac{2\pi}{(2\pi)^a}\int_0^\infty e^{-2\pi x}(2\pi x)^{a-1}\,dx = \frac{1}{(2\pi)^a}\int_0^\infty e^{-t}t^{a-1}\,dt = \frac{\Gamma(a)}{(2\pi)^a}$$

and hence $f \in L^1$. Similarly, if $a > \frac{1}{2}$ then $f \in L^2$ because

$$\int |f(x)|^2\,dx = \int_0^\infty e^{-4\pi x}x^{2a-2}\,dx = \frac{(4\pi)^2}{(4\pi)^{2a}}\int_0^\infty e^{-4\pi x}(4\pi x)^{2a-2}\,dx = \frac{4\pi}{(4\pi)^{2a}}\int_0^\infty e^{-t}t^{2a-2}\,dt = \frac{\Gamma(2a-1)}{(4\pi)^{2a-1}}.$$

(b) Given $\xi \in \mathbb{R}$, define $\gamma : \mathbb{R} \to \mathbb{C}$ by $\gamma(x) := 2\pi(i\xi + 1)x$. Note that

$$\widehat{f}(\xi) = \int e^{-2\pi i\xi x}f(x)\,dx = \int_0^\infty e^{-2\pi(i\xi+1)x}x^{a-1}\,dx = \frac{2\pi(i\xi+1)}{(2\pi(i\xi+1))^a}\int_0^\infty e^{-2\pi(i\xi+1)x}(2\pi(i\xi+1)x)^{a-1}\,dx.$$

By part (a) $x \mapsto e^{-2\pi i\xi x}f(x)$ is in L^1, so the dominated convergence theorem implies that

$$\begin{aligned}\int_0^\infty e^{-2\pi(i\xi+1)x}(2\pi(i\xi+1)x)^{a-1}\,dx &= \lim_{N\to\infty}\int_0^N e^{-2\pi(i\xi+1)x}(2\pi(i\xi+1)x)^{a-1}\,dx \\ &= \frac{1}{2\pi(i\xi+1)}\lim_{N\to\infty}\int_{\gamma|_{[0,N]}} e^{-z}z^{a-1}\,dz.\end{aligned}$$

Given $N \in \mathbb{N}$, define $\gamma_1 : [0, 2\pi N] \to \mathbb{C}$ by $\gamma_1(x) := x$ and $\gamma_2 : [0 : 2\pi N]$ by $\gamma_2(x) := 2\pi N + i\xi x$. Clearly γ_1 followed by γ_2 is homotopic to $\gamma|_{[0,N]}$, so by Cauchy's theorem

$$\int_{\gamma|_{[0,N]}} e^{-z}z^{a-1}\,dz = \int_{\gamma_1} e^{-z}z^{a-1}\,dz + \int_{\gamma_2} e^{-z}z^{a-1}\,dz.$$

Note that $\lim_{N\to\infty}\int_{\gamma_1} e^{-z}z^{a-1}\,dz = \lim_{N\to\infty}\int_0^{2\pi N} e^{-x}x^{a-1}\,dx = \Gamma(a)$, whereas

$$\left|\int_{\gamma_2} e^{-z}z^{a-1}\,dz\right| \leq \int_0^{2\pi N} |e^{-(2\pi N+i\xi x)}(2\pi N+i\xi x)^{a-1}i\xi|\,dx = e^{-2\pi N}|\xi|\int_0^{2\pi N} |2\pi N+i\xi x|^{a-1}\,dx.$$

If $a \geq 1$ then $|2\pi N+i\xi x|^{a-1} \leq (2\pi N+|\xi|x)^{a-1} \leq (2\pi N(1+|\xi|))^{a-1}$ for all $x\in[0,2\pi N]$. Otherwise $|2\pi N + i\xi x|^{a-1} = (4\pi^2N^2+|\xi|^2x^2)^{(a-1)/2} \leq (2\pi N)^{a-1}$ for all $x\in[0,2\pi N]$. In either case, the above integral exhibits polynomial growth as $N\to\infty$ (which is slower than exponential decay), so

$$\lim_{N\to\infty}\int_{\gamma|_{[0,N]}} e^{-z}z^{a-1}\,dz = \lim_{N\to\infty}\int_{\gamma_1} e^{-z}z^{a-1}\,dz + \lim_{N\to\infty}\int_{\gamma_2} e^{-z}z^{a-1}\,dz = \Gamma(a)+0=\Gamma(a).$$

Combining the above calculations, it follows that $\widehat{f}(\xi) = (2\pi(i\xi+1))^{-a}\Gamma(a)$.

(c) Define $g:\mathbb{R}\to\mathbb{R}$ in a similar manner to f but with b instead of a. By part (a) both f and g are in L^2, and

$$\begin{aligned}\int(1-ix)^{-a}(1+ix)^{-b}\,dx &= (2\pi)^{a+b}\int\overline{(2\pi(1+ix))^{-a}}(2\pi(1+ix))^{-b}\,dx\\ &= (2\pi)^{a+b}\int\overline{\widehat{f}(x)\Gamma(a)^{-1}}\widehat{g}(x)\Gamma(b)^{-1}\,dx\\ &= \frac{(2\pi)^{a+b}}{\Gamma(a)\Gamma(b)}(\widehat{g},\widehat{f})\end{aligned}$$

by part (b). Since the Fourier transform is unitary (from L^2 to L^2),

$$(\widehat{g},\widehat{f}) = (g,f) = \int\overline{f(x)}g(x)\,dx = \int_0^\infty e^{2\pi x}x^{a-1}e^{2\pi x}x^{b-1}\,dx = \int_0^\infty (e^{2\pi x})^2x^{a+b-2}\,dx = \int |h(x)|^2\,dx$$

where $h:\mathbb{R}\to\mathbb{R}$ is defined in a similar manner to f but with $\frac{a+b}{2}$ instead of a. By part (a), it follows that

$$\int(1-ix)^{-a}(1+ix)^{-b}\,dx = \frac{(2\pi)^{a+b}\Gamma(a+b-1)}{(4\pi)^{a+b-1}\Gamma(a)\Gamma(b)} = \frac{2^{2-a-b}\pi\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}.$$

19. Since $\{x\in\mathbb{R}^n \mid f(x)\neq 0\}$ has finite measure, $f\in L^1$ and hence

$$\widehat{f}(\xi) = \int e^{-2\pi i\xi\cdot x}f(x)\,dx$$

for all $\xi\in\mathbb{R}^n$. Given a measurable set $E\subseteq\mathbb{R}^n$, the Minkowski and Hölder inequalities imply that

$$\begin{aligned}\int_E|\widehat{f}|^2 &= \int_E\left|\int e^{-2\pi i\xi\cdot x}f(x)\,dx\right|^2 d\xi\\ &= \int\left|\int e^{-2\pi i\xi\cdot x}f(x)\sqrt{\chi_E(\xi)}\,dx\right|^2 d\xi\\ &\leq \left(\int\sqrt{\int|e^{-2\pi i\xi\cdot x}f(x)\chi_E(\xi)|^2\,d\xi}\,dx\right)^2\\ &= \left(\int|f(x)|\sqrt{\int\chi_E(\xi)\,d\xi}\,dx\right)^2\\ &= \left(\sqrt{m(E)}\int_S|f(x)|\,dx\right)^2\\ &\leq m(E)(\|f\|_2\|\chi_S\|_2)^2\\ &= \|f\|_2^2m(S)m(E).\end{aligned}$$

22. (a) Let $\xi, \zeta \in \mathbb{R}^n$ and suppose that $|\xi| = |\zeta|$. If $|\xi| = 0$ then $\xi = 0 = \zeta$ and hence $J(\xi) = J(\zeta)$. Otherwise, set $\alpha := |\xi|^{-1}$, and note that $\{\alpha\xi\}$ and $\{\alpha\zeta\}$ can be extended to orthonormal bases for $\mathbb{R}^n$. Let $R \in O(n)$ be the change of basis matrix between these bases, and note that $R(\alpha\xi) = \alpha\zeta$, implying that $R\xi = \zeta$. It follows that $J(\zeta) = J(R\xi) = \int_S e^{ix\cdot(R\xi)}\,d\sigma(x) = \int_S e^{i(R^{-1}x)\cdot\xi}\,d\sigma(x) = |\det(R^{-1})|^{-1}\int_S e^{ix\cdot\xi}\,d\sigma(x) = J(\xi)$. Therefore $J(\xi)$ depends only on $|\xi|$. By Theorem 2.49

$$\begin{aligned}\widehat{F}(\xi) &= \int e^{-2\pi i\xi\cdot x} f(|x|)\,dx \\ &= \int_0^\infty \int_S e^{-2\pi i\xi\cdot(rx)} f(|rx|) r^{n-1}\,d\sigma(x)\,dr \\ &= \int_0^\infty \int_S e^{i(2\pi r\xi)\cdot(-x)}\,d\sigma(x) f(r) r^{n-1}\,dr \\ &= \int_0^\infty |-1|^{-1} J(2\pi r\xi) f(r) r^{n-1}\,dr \\ &= \int_0^\infty j(2\pi r|\xi|) f(r) r^{n-1}\,dr,\end{aligned}$$

so ($\widehat{F}$ is radial and) $g(s) = \int_0^\infty j(2\pi rs) f(r) r^{n-1}\,dr$ for all $s \in [0,\infty)$.

(b) If $k \in \{1,\ldots,n\}$ then $\frac{\partial}{\partial\xi_k} e^{ix\cdot\xi} = ix_k e^{ix\cdot\xi}$ and hence $|\frac{\partial}{\partial\xi_k} e^{ix\cdot\xi}| = |x_k| \le 1$ for all $x \in S$ and $\xi \in \mathbb{R}^n$. Moreover $\frac{\partial^2}{\partial\xi_k^2} e^{ix\cdot\xi} = (ix_k)^2 e^{ix\cdot\xi} = -x_k^2 e^{ix\cdot\xi}$, so $|\frac{\partial^2}{\partial\xi_k^2} e^{ix\cdot\xi}| = |x_k|^2 \le 1$ for all $x \in S$ and $\xi \in \mathbb{R}^n$. Since $\int 1\,d\sigma = \sigma(S) < \infty$, the dominated convergence theorem implies that

$$(\partial_k^2 J)(\xi) = \frac{\partial}{\partial\xi_k}\int_S \frac{\partial}{\partial\xi_k} e^{ix\cdot\xi}\,d\sigma(x) = \int_S \frac{\partial^2}{\partial\xi_k^2} e^{ix\cdot\xi}\,d\sigma(x) = \int_S -x_k^2 e^{ix\cdot\xi}\,d\sigma(x)$$

for all $\xi \in \mathbb{R}^n$, in which case $\sum_{k=1}^n \partial_k^2 J + J = 0$. Indeed, if $\xi \in \mathbb{R}^n$ then

$$\sum_{k=1}^n (\partial_k^2 J)(\xi) = \int_S \left(\sum_{k=1}^n -x_k^2 e^{ix\cdot\xi}\right) d\sigma(x) = -\int_S \left(\sum_{k=1}^n x_k^2\right) e^{ix\cdot\xi}\,d\sigma(x) = -\int_S |x|^2 e^{ix\cdot\xi}\,d\sigma(x) = -J(\xi).$$

(c) If $k \in \{1,\ldots,k\}$ and $\xi \in \mathbb{R}^n$ is non-zero then $\frac{\partial}{\partial\xi_k}|\xi| = \frac{\partial}{\partial\xi_k}(\xi\cdot\xi)^{1/2} = \frac{1}{2}(\xi\cdot\xi)^{-1/2}(2\xi_k) = \xi_k|\xi|^{-1}$, so that

$$(\partial_k J)(\xi) = \frac{\partial}{\partial\xi_k} j(|\xi|) = j'(|\xi|)\frac{\xi_k}{|\xi|}$$

and hence

$$(\partial_k^2 J)(\xi) = \frac{\partial}{\partial\xi_k}\left(j'(|\xi|)\frac{\xi_k}{|\xi|}\right) = j''(|\xi|)\frac{\xi_k^2}{|\xi|^2} + j'(|\xi|)\frac{|\xi| - \xi_k^2|\xi|^{-1}}{|\xi|^2}.$$

This implies that

$$\begin{aligned}\sum_{k=1}^n \partial_k^2 J(\xi) + J(\xi) &= j''(|\xi|)\sum_{k=1}^n \frac{\xi_k^2}{|\xi|^2} + j'(|\xi|)\sum_{k=1}^n \frac{|\xi|}{|\xi|^2} - j'(|\xi|)\sum_{k=1}^n \frac{\xi_k^2}{|\xi|^3} + j(|\xi|) \\ &= j''(|\xi|) + j'(|\xi|)\left(\frac{n}{|\xi|} - \frac{1}{|\xi|}\right) + j(|\xi|) \\ &= j''(|\xi|) + (n-1)j'(|\xi|)|\xi|^{-1} + j(|\xi|),\end{aligned}$$

which is zero by part (b), in which case $|\xi|j''(|\xi|) + (n-1)j'(|\xi|) + |\xi|j(|\xi|) = 0$, for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Moreover $(\partial_1 J)(0,\ldots,0) = j'(0) = -(\partial_1 J)(0,\ldots,0)$, since $J(r,0,\ldots,0) = j(r) = J(-r,0,\ldots,0)$ for all $r \in [0,\infty)$, so $j'(0) = 0$ and hence $rj''(r) + (n-1)j'(r) + rj(r) = 0$ for all $r \in [0,\infty)$.

(d) Define $f : [0,\infty) \to \mathbb{C}$ by $f(r) := rj(r)$, so that $f'(r) = j(r) + rj'(r)$ and hence $f''(r) = 2j'(r) + rj''(r)$ for all $r \in [0,\infty)$. It follows from part (c) that $f'' + f = 0$, which is a second order ODE satisfied by the function $g : [0,\infty) \to \mathbb{R}$ defined by $g(x) := \sigma(S)\sin(x)$. Since $f'(0) = j(0) = \int_S 1\, d\sigma = \sigma(S) = g'(0)$ and $f(0) = 0 = g(0)$, it follows that $f = g$, in which case $j(r) = r^{-1}f(r) = \sigma(S)r^{-1}\sin(r) = 4\pi r^{-1}\sin(r)$ for all $r \in (0,\infty)$. Since $j(0) = \sigma(S) = 4\pi$, this statement can be interpreted for $r = 0$ by taking limits.

23. (a) Define linear operators P, Q on $\mathcal{S}(\mathbb{R})$ by $Pf(x) := f'(x)$ and $Qf(x) := xf(x)$. If $f, g \in S(\mathbb{R})$ then

$$\int (Qf)\overline{g} = \int xf(x)\overline{g(x)}\, dx = \int f(x)\overline{xg(x)}\, dx = \int f(\overline{Qg}).$$

Moreover, $(f\overline{g})' = f'\overline{g} + f\overline{g'}$, and (by the monotone convergence theorem applied separately to the positive and negative real and imaginary parts of $f'\overline{g}$, followed by the fundamental theorem of calculus)

$$\int (f\overline{g})' = \lim_{N\to\infty}\int_{-N}^{N}(f\overline{g})' = \lim_{N\to\infty}(f(N)\overline{g(N)} - f(-N)\overline{g(-N)}) = 0.$$

It follows that

$$\int (Pf)\overline{g} = \int f'\overline{g} = \int (f\overline{g})' - \int f\overline{g'} = -\int f(\overline{Pg}),$$

in which case

$$\sqrt{2}\int (Tf)\overline{g} = \int (Qf - Pf)\overline{g} = \int (Qf)\overline{g} - \int (Pf)\overline{g} = \int f(\overline{Qg}) + \int f(\overline{Pg}) = \int f(\overline{Qg + Pg}) = \sqrt{2}\int f(\overline{T^*g}).$$

Thus $\int (Tf)\overline{g} = \int f(\overline{T^*g})$. If $x \in \mathbb{R}$ then $(PQf)(x) = (Qf)'(x) = f(x) + xf'(x) = f(x) + (QPf)(x)$, so

$$2[T^*, T] = [Q + P, Q - P] = [Q, Q] - [Q, P] + [P, Q] - [P, P] = 2[P, Q] = 2I.$$

This implies that $[T^*, T] = I = T^0$. If $k \in \mathbb{N}$ with $k > 1$ and $[T^*, T^{k-1}] = (k-1)T^{k-2}$, then

$$T^*T^k = T^*T^{k-1}T = [T^*, T^{k-1}]T + T^{k-1}T^*T = (k-1)T^{k-1} + T^{k-1}[T^*, T] + T^{k-1}TT^* = kT^{k-1} + T^kT^*,$$

in which case $[T^*, T^k] = kT^{k-1}$. By induction, this shows that $[T^*, T^k] = kT^{k-1}$. for all $k \in \mathbb{N}$.

(b) If $k \in \mathbb{N}$ then $Th_k = (k!)^{-1/2}T^{k+1}h_0 = \sqrt{k+1}((k+1)!)^{-1/2}T^{k+1}h_0 = \sqrt{k+1}h_{k+1}$, and since

$$\sqrt{2}(T^*h_0)(x) = xh_0(x) + h_0'(x) = \pi^{-1/4}xe^{-x^2/2} + \pi^{-1/4}(-x)e^{-x^2/2} = 0,$$

part (a) implies that $T^*h_k = (k!)^{-1/2}T^*T^kh_0 = (k!)^{-1/2}([T^*, T^k] + T^kT^*)h_0 = (k!)^{-1/2}kT^{k-1}h_0 = \sqrt{k}h_{k-1}$. These formulae also hold for $k = 0$ if we let h_{-1} be an arbitrary function, and it clearly follows that $TT^*h_k = T(\sqrt{k}h_{k-1}) = \sqrt{k}Th_{k-1} = \sqrt{k}\sqrt{k}h_k = kh_k$ for all $k \in \mathbb{N}\cup\{0\}$.

(c) Note that $S = 2TT^* + I = (Q - P)(Q + P) + I = Q^2 + [Q, P] - P^2 + [P, Q] = Q^2 - P^2$. Hence, if $f \in \mathcal{S}(\mathbb{R})$ then $Sf(x) = Q^2f(x) - P^2f(x) = x^2f(x) - f''(x)$ for all $x \in \mathbb{R}$. Moreover if $k \in \mathbb{N}\cup\{0\}$ then

$$Sh_k = 2TT^*h_k + h_k = 2kh_k + h_k = (2k+1)h_k.$$

(d) Note that $\|h_0\|_2^2 = \int |h_0|^2 = \int \frac{e^{-x^2}}{\sqrt{\pi}}\, dx = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$. If $k \in \mathbb{N}$ and $\|h_{k-1}\|_2 = 1$, then

$$\|h_k\|_2^2 = \int h_k\overline{h_k} = k^{-1}\int (TT^*h_k)\overline{h_k} = k^{-1}\int (T^*h_k)(\overline{T^*h_k}) = k^{-1}\int \sqrt{k}h_{k-1}\overline{\sqrt{k}h_{k-1}} = \int |h_{k-1}|^2 = 1,$$

so by induction $\|h_k\| = 1$ for all $k \in \mathbb{N} \cup \{0\}$. If $j, k \in \mathbb{N} \cup \{0\}$ and $j > k$ then

$$(h_j, h_k) = \int h_j \overline{h_k} = j^{-1} \int (TT^* h_j)\overline{h_k} = j^{-1} \int (T^* h_j)(\overline{T^* h_k}) = j^{-1} \int \sqrt{j} h_{j-1}(\overline{\sqrt{k} h_{k-1}}) = \sqrt{\frac{k}{j}}(h_{j-1}, h_{k-1})$$

so by induction $(h_j, h_k) = \sqrt{\frac{k(k-1)\cdots 0}{j(j-1)\cdots(j-k)}}(h_{j-k-1}, h_{-1}) = 0$. This shows that $\{h_k\}_{k=0}^\infty$ is orthonormal.

(e) If $k \in \mathbb{N}$ and

$$T^{k-1} f(x) = (-1)^{k-1} 2^{(1-k)/2} e^{x^2/2} \frac{d^{k-1}}{dx^{k-1}}(e^{-x^2/2} f(x)),$$

for all $x \in \mathbb{R}$ (which is clearly true when $k = 1$), the product rule implies that

$$\begin{aligned} T^k f(x) &= (-1)^{k-1} 2^{-k/2} \left(x e^{x^2/2} \frac{d^{k-1}}{dx^{k-1}}(e^{-x^2/2} f(x)) - x e^{x^2/2} \frac{d^{k-1}}{dx^{k-1}}(e^{-x^2/2} f(x)) - e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2} f(x)) \right) \\ &= (-1)^k 2^{-k/2} e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2} f(x)). \end{aligned}$$

for all $x \in \mathbb{R}$. Therefore $T^k f(x) = (-1)^k 2^{-k/2} e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2} f(x))$ for all $x \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$, by induction. If $k \in \mathbb{N} \cup \{0\}$ then $h_k = k^{-1/2} T h_{k-1} = \cdots = (k!)^{-1/2} T^k h_0$, so for all $x \in \mathbb{R}$

$$h_k(x) = (k!)^{-1/2}(-1)^k 2^{-k/2} e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2} h_0(x)) = \frac{(-1)^k}{\sqrt{\pi^{1/2} 2^k k!}} e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}.$$

(f) Given $k \in \mathbb{N} \cup \{0\}$, it is easily shown by induction and the product rule that $\frac{d^k}{dx^k} e^{-x^2} = P_k(x) e^{-x^2}$ for all $x \in \mathbb{R}$, where $P_k(x)$ is some polynomial of degree k. The formula for h_k from part (e) implies that

$$H_k(x) = \frac{(-1)^k}{\sqrt{\pi^{1/2} 2^k k!}} e^{x^2} P_k(x) e^{-x^2} = \frac{(-1)^k}{\sqrt{\pi^{1/2} 2^k k!}} P_k(x),$$

so $H_k(x)$ is also a polynomial of degree k. In particular $H_0(x)$ is a non-zero constant, so all the constant polynomials are in $\operatorname{span}\{H_0(x)\}$. If the polynomials of degree less than k are in $\operatorname{span}\{H_j(x)\}_{j=0}^{k-1}$ and $c \in \mathbb{R} \setminus \{0\}$ is the leading term of $H_k(x)$, then $x^n - c^{-1} H_k(x) \in \operatorname{span}\{H_j(x)\}_{j=0}^{k-1}$, which implies that $x^n \in \operatorname{span}\{H_j(x)\}_{j=0}^k$ and hence every polynomial of degree at most k is in $\operatorname{span}\{H_j(x)\}_{j=0}^k$. By induction, this shows that $\operatorname{span}\{H_j(x)\}_{j=0}^k$ is the set of polynomials of degree at most k, for all $k \in \mathbb{N} \cup \{0\}$.

(g) Let $f \in L^2$ and suppose that $f \perp h_k$ for all $k \in \mathbb{N} \cup \{0\}$. Define $g : \mathbb{R} \to \mathbb{C}$ by $g(x) := f(x) e^{-x^2/2}$, so that $g \in L^1$ (by Hölder's inequality). If $\xi, x \in \mathbb{R}$ and $N \in \mathbb{N}$ then

$$\left| \sum_{k=0}^N \frac{(-2\pi i \xi x)^k}{k!} g(x) \right| \leq \sum_{k=0}^N \frac{|2\pi \xi x|^k}{k!} |f(x)| e^{-x^2/2} \leq e^{2\pi|\xi x| - x^2/2} |f(x)|.$$

If $\xi \in \mathbb{R}$ then $x \mapsto e^{2\pi|\xi x| - x^2/2}$ is clearly in L^2, so $x \mapsto e^{2\pi|\xi x| - x^2/2}|f(x)|$ is in L^1. Moreover

$$\widehat{g}(\xi) = \int e^{-2\pi i \xi x} g(x)\, dx = \int \sum_{k=0}^\infty \frac{(-2\pi i \xi x)^k}{k!} g(x)\, dx = \lim_{N \to \infty} \int \sum_{k=0}^N \frac{(-2\pi i \xi x)^k}{k!} g(x)\, dx$$

by the dominated convergence theorem. If $N \in \mathbb{N}$ then $\sum_{k=0}^N \frac{(2\pi i \xi x)^k}{k!} \in \operatorname{span}_{\mathbb{C}}\{H_k(x)\}_{k=0}^N$, and since $\overline{H_k} g = f \overline{h_k}$ for all $k \in \mathbb{N} \cup \{0\}$, it follows that $\widehat{g}(\xi) = \lim_{N \to \infty} 0 = 0$. In particular $\widehat{g} \in L^1$, so by the Fourier inversion theorem $g = (\widehat{g})^\vee = 0$ almost everywhere. Since $e^{-x^2/2} > 0$ for all $x \in \mathbb{R}$, this implies that $f = 0$ in L^2. Therefore $\{h_k\}_{k=0}^\infty$ is an orthonormal basis for L^2.

(h) Clearly A is linear and bijective (its inverse is given by $A^{-1}f(x) := (2\pi)^{-1/4}f((2\pi)^{-1/2}x)$). If $f \in L^2$ then

$$\|Af\|_2^2 = \int |Af(x)|^2\,dx = \int \sqrt{2\pi}|f(x\sqrt{2\pi})|^2\,dx = \frac{1}{\sqrt{2\pi}}\int \sqrt{2\pi}|f(t)|^2\,dt = \|f\|_2^2,$$

which shows that A is unitary (by the polarisation identity). Moreover, if $\xi \in \mathbb{R}$ then (assuming $f \in L^1$)

$$\widehat{Af}(\xi) = \int e^{-2\pi i\xi x}(2\pi)^{1/4}f(x\sqrt{2\pi})\,dx = (2\pi)^{1/4}\int e^{-\sqrt{2\pi}i\xi x\sqrt{2\pi}}f(x\sqrt{2\pi})\,dx = \frac{1}{(2\pi)^{1/4}}\int e^{-\sqrt{2\pi}i\xi t}f(t)\,dt,$$

in which case

$$\widetilde{f}(\xi) = A^{-1}\widehat{Af}(\xi) = \frac{(2\pi)^{-1/4}}{(2\pi)^{1/4}}\int e^{-\sqrt{2\pi}i(2\pi)^{-1/2}\xi t}f(t)\,dt = \frac{1}{\sqrt{2\pi}}\int e^{-i\xi t}f(t)\,dt.$$

If $f \in \mathcal{S}$ then clearly $\widetilde{f} \in S$, and it follows that

$$\sqrt{2\pi}\widetilde{Tf}(\xi) = \int e^{-i\xi t}Tf(t)\,dt = \int Tf(t)\overline{e^{i\xi t}}\,dt = \int f(t)\frac{\overline{(te^{i\xi t} + i\xi e^{i\xi t})}}{\sqrt{2}}\,dt = \frac{1}{\sqrt{2}}\int f(t)(t - i\xi)e^{-i\xi t}\,dt.$$

On the other hand

$$-\sqrt{2\pi}iT(\widetilde{f})(\xi) = -\frac{i}{\sqrt{2}}\left(\xi\int e^{-i\xi t}f(t)\,dt - \frac{d}{d\xi}\int e^{-i\xi t}f(t)\,dt\right).$$

Since $|\frac{d}{d\xi}e^{-i\xi t}f(t)| = |-ite^{-i\xi t}f(t)| = |tf(t)|$ for all $t \in \mathbb{R}$, and $t \mapsto tf(t)$ is in L^1 (as $f \in \mathcal{S}$), it follows that

$$\begin{aligned}
-\sqrt{2\pi}iT(\widetilde{f})(\xi) &= -\frac{i}{\sqrt{2}}\left(\xi\int e^{-i\xi t}f(t)\,dt - \int \frac{d}{d\xi}e^{-i\xi t}f(t)\,dt\right)\\
&= -\frac{i}{\sqrt{2}}\left(\int \xi e^{-i\xi t}f(t)\,dt + \int ite^{-i\xi t}f(t)\,dt\right)\\
&= \frac{1}{\sqrt{2}}\left(\int te^{-i\xi t}f(t)\,dt - \int i\xi e^{-i\xi t}f(t)\,dt\right)\\
&= \frac{1}{\sqrt{2}}\int (t - i\xi)e^{-i\xi t}f(t)\,dt\\
&= \sqrt{2\pi}\widetilde{Tf}(\xi).
\end{aligned}$$

In particular $\widetilde{Tf} = -iT(\widetilde{f})$. Note that $Ah_0(x) = (2\pi)^{1/4}h_0(\sqrt{2\pi}x) = 2^{1/4}e^{-\pi x^2}$ for all $x \in \mathbb{R}$ and hence $\widehat{Ah_0}(\xi) = 2^{1/4}e^{-\pi\xi^2}$ for all $\xi \in \mathbb{R}$ (by Proposition 8.24). It follows that

$$\widetilde{h}_0(\xi) = A^{-1}\widehat{Ah_0}(\xi) = (2\pi)^{-1/4}2^{1/4}e^{-\pi(2\pi)^{-1}\xi^2} = \pi^{-1/4}e^{\xi^2/2} = h_0(\xi)$$

for all $\xi \in \mathbb{R}$. If $k \in \mathbb{N}$ then $h_k = (k!)^{-1/2}T^k h_0$, so for all $\xi \in \mathbb{R}$

$$\widetilde{h}_k(\xi) = (k!)^{-1/2}\widetilde{T^k h_0}(\xi) = (k!)^{-1/2}(-i)T(\widetilde{T^{k-1}h_0})(\xi) = \cdots = (k!)^{-1/2}(-i)^k T^k\widetilde{h}_0(\xi) = (-i)^k h_k(\xi).$$

Since $\{h_k\}_{k=1}^\infty$ is an orthonormal basis for L^2, its unitary image $\{\phi_k\}_{k=0}^\infty$ is also an orthonormal basis for L^2. Moreover, for each $k \in \mathbb{N} \cup \{0\}$ it is clear that $\widehat{\phi}_k = \widehat{Ah_k} = AA^{-1}\widehat{Ah_k} = A\widetilde{h}_k = A(-i)^k h_k = (-i)^k\phi_k$.

25. (a) Let $f \in AC(\mathbb{R})$ be periodic and suppose that $f' \in L^p(\mathbb{T})$. If $x, y \in \mathbb{T}$ and $x > y$ then

$$|f(x) - f(y)| = \left|\int_y^x f'\right| \le \int_y^x |f'| \le \|f'\|_p\|\chi_{[y,x]}\|_q = \|f'\|_p(x - y)^{1/q},$$

in which case $|f(x) - f(y)| \leq \|f'\|_p |x-y|^{1/q}$ for all $x, y \in \mathbb{R}$, because f is periodic. Therefore $f \in \Lambda_{1/q}(\mathbb{T})$. Given $\alpha \in (\frac{1}{q}, 1)$, choose $\beta \in (\frac{1}{q}, \alpha)$ and define $g : [0, \frac{1}{2}] \to \mathbb{R}$ by $g(x) := \beta x^{\beta - 1}$. Note that $g \in L^p([0, \frac{1}{2}])$ because $(\beta - 1)p > (\frac{1}{q} - 1)p = -1$. Now define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \int_0^x \sum_{k \in \mathbb{Z}} ((\tau_k g)\chi_{[k.k+\frac{1}{2})} - (\tau_{k+\frac{1}{2}} g)\chi_{[k+\frac{1}{2}, k+1)}),$$

so that, by definition, f is periodic and absolutely continuous on every bounded interval. Moreover

$$\frac{|f(x) - f(0)|}{|x - 0|^\alpha} = \frac{1}{x^\alpha} \int_0^x g = \frac{x^\beta - 0^\beta}{x^\alpha} = x^{\beta - \alpha}$$

for all $x \in (0, \frac{1}{2})$, and $\lim_{x \searrow 0} x^{\beta - \alpha} = \infty$, so $f \notin \Lambda_\alpha(\mathbb{T})$.

(b) Let $\alpha \in (0, 1)$ and define $h : [0, 1] \to [0, 1]$ by $h(t) := \sin(\pi t)$. Since $\alpha < 1$ the series $\sum_{k=1}^\infty k^{-1/\alpha}$ converges. Let $S \in (0, \infty)$ be its sum and for each $n \in \mathbb{N}$ set $t_n := S^{-1} \sum_{k=1}^{n-1} k^{-1/\alpha}$. Now define $f : [0, 1] \to [0, 1]$ by

$$f(t) := \sum_{n=1}^\infty \frac{\chi_{[t_n, t_{n+1}]}(t)}{n} h\left(\frac{x - t_n}{t_{n+1} - t_n}\right).$$

If $x \in [0, 1]$ then $x \in [t_n, t_{n+1}]$ for some $n \in \mathbb{N}$. Given $y \in [t_n, t_{n+1}]$, the mean value theorem implies that

$$\begin{aligned}
|f(x) - f(y)| &= \frac{1}{n} \left| h\left(\frac{x - t_n}{t_{n+1} - t_n}\right) - h\left(\frac{y - t_n}{t_{n+1} - t_n}\right) \right| \\
&\leq \frac{1}{n} \left| \frac{x - t_n}{t_{n+1} - t_n} - \frac{y - t_n}{t_{n+1} - t_n} \right| \\
&= \frac{|x - y|}{n|t_{n+1} - t_n|} \\
&\leq \frac{|x - y|^\alpha |t_{n+1} - t_n|^{1-\alpha}}{n|t_{n+1} - t_n|} \\
&= \frac{|x - y|^\alpha}{n|t_{n+1} - t_n|^\alpha} \\
&= \frac{|x - y|^\alpha}{n(S^{-1} n^{-1/\alpha})^\alpha} \\
&= S^\alpha |x - y|^\alpha.
\end{aligned}$$

Given $y \in [t_{n+1}, t_{n+2}]$, set $z := t_{n+1}$ and note that

$$|f(x) - f(y)|^{1/\alpha} \leq 2^{1/\alpha}(|f(x) - f(z)|^{1/\alpha} + |f(z) - f(y)|^{1/\alpha}) \leq 2^{1/\alpha}(S|x - z| + S|z - y|),$$

so $|f(x) - f(y)| \leq 2S^\alpha(|x - z| + |z - y|)^\alpha = 2S^\alpha(z - x + y - z)^\alpha = 2S^\alpha|x - y|^\alpha$. If $y \in [t_{n+2}, 1]$ then

$$S^\alpha|x - y|^\alpha = (S(y - x))^\alpha \geq (S(t_{n+2} - t_{n-1}))^\alpha = \frac{1}{n+1} \geq \frac{1}{2n} \geq \frac{|f(x)| + |f(y)|}{4} \geq \frac{|f(x) - f(y)|}{4}.$$

This shows that $|f(x) - f(y)| \leq 4S^\alpha|x - y|^\alpha$ for all $x, y \in [0, 1]$, in which case $f \in \Lambda_\alpha(\mathbb{T})$. However

$$\sum_{n=1}^N \left| f\left(\frac{t_n + t_{n+1}}{2}\right) - f(t_n) \right| = \sum_{n=1}^N \left| \frac{1}{n} - 0 \right| = \sum_{n=1}^N \frac{1}{n}$$

for all $N \in \mathbb{N}$, so f is not of bounded variation.

30. Define $\Phi : \mathbb{R}^n \to \mathbb{R}$ by $\Phi(\xi) := e^{-\pi|\xi|^2}$, so that $\Phi(0) = 1$ and $\Phi^\vee = \Phi \in L^1$. Note that

$$2n\log(1+|\xi|) \le 2|\xi| \int_1^{1+|\xi|} \frac{dt}{t} \le 2|\xi| \int_1^{1+|\xi|} dt = 2|\xi|^2 \le \pi|\xi|^2 + 2n\log(1+n)$$

for all $\xi \in (-\infty, -n) \cup (n, \infty)$. Moreover $2n\log(1+|\xi|) \le 2n\log(1+n) \le \pi|\xi|^2 + 2n\log(1+n)$ for all $\xi \in [-n, n]$. It follows that $(1+|\xi|)^{2n} \le Ce^{\pi|\xi|^2}$, where $C := (1+n)^{2n}$, and hence $|\Phi(\xi)| = \Phi(\xi) \le C(1+|\xi|)^{-2n}$, for all $\xi \in \mathbb{R}$. Since $0 \in L_f$, Theorem 8.35 implies that

$$\lim_{t\searrow 0} \int \widehat{f}(\xi)e^{-\pi t^2|\xi|^2}\,d\xi = \lim_{t\searrow 0} \int \widehat{f}(\xi)\Phi(t\xi)e^{2\pi i\xi\cdot 0}\,d\xi = f(0),$$

in which case, by Fatou's lemma,

$$\|\widehat{f}\|_1 = \int \widehat{f}(\xi)\,d\xi = \int \lim_{t\searrow 0} \widehat{f}(\xi)e^{-\pi t^2|\xi|^2}\,d\xi \le f(0) < \infty.$$