1. (a) By Hölder's inequality, if $\phi \in C_c^\infty(U)$ then integration against ϕ is an element of $(L^p)^*$. Since convergence in L^p implies weak convergence, $\lim_{n\to\infty} \int f_n\phi = \int f\phi$. This shows that $(f_n)_{n=1}^\infty$ converges to f in $\mathcal{D}'(U)$.

   (b) If $\phi \in C_c^\infty(U)$ then the support of ϕ is bounded, and ϕ itself is bounded, so $g|\phi| \in L^1$. Since $|f_n\phi| \le g|\phi|$ for all $n \in \mathbb{N}$, the dominated convergence theorem implies that $\lim_{n\to\infty} \int f_n\phi = \int f\phi$.

   (c) The sequence $(n\chi_{(0,\frac{1}{n})})_{n=1}^\infty$ converges to 0 pointwise, but to $\delta \ne 0$ in $\mathcal{D}'(\mathbb{R})$ (by Proposition 9.1).

10. By Theorem 2.49, if $R \in (0,\infty)$ and $f \ne 0$ then

$$\int_{B_R(0)} |f| = \int_0^R \int_{S^{n-1}} |f(rx)|r^{n-1}\,d\sigma(x)\,dr = \int_{S^{n-1}} \int_0^R |f(x)|r^{-1}\,dr\,d\sigma(x) = \int_{S^{n-1}} |f(x)|\cdot\infty\,d\sigma(x) = \infty,$$

so f is not locally integrable near 0. If $\phi \in C_c^\infty$ is supported on $B_R(0)$ and $\varepsilon \in (0,R)$ then

$$\int_{B_\varepsilon(0)^c} f\phi = \int_\varepsilon^R \int_{S^{n-1}} f(rx)\phi(rx)r^{n-1}\,d\sigma(x)\,dr = \int_\varepsilon^R \int_{S^{n-1}} f(x)(\phi(rx)-\phi(0))r^{-1}\,d\sigma(x)\,dr = \int_{B_R(0)\setminus B_\varepsilon(0)} f(\phi-\phi(0))$$

because $\int_{S^{n-1}} f\,d\sigma = 0$. By the mean value theorem there exists $M \in (0,\infty)$ such that $|\phi(x)-\phi(0)| \le M|x|$ for all $x \in \mathbb{R}^n$, in which case

$$\int_{B_R(0)} |f||\phi-\phi(0)| \le \int_0^R \int_{S^{n-1}} |f(rx)|M|rx|r^{n-1}\,d\sigma(x)\,dr = \int_{S^{n-1}} \int_0^R |f(x)|M\,dr\,d\sigma(x) = MR\int_{S^{n-1}} |f|\,d\sigma,$$

so by the dominated convergence theorem $\langle PV(f),\phi\rangle = \int_{B_R(0)} f(\phi-\phi(0))$. In particular $PV(f)$ is well-defined. It is clearly linear. If $K \subseteq B_R(0)$ is compact and $(\phi_n)_{n=1}^\infty$ a sequence in $C^\infty(K)$ which converges to $\phi \in C^\infty(K)$, then

$$\lim_{n\to\infty} \langle PV(f),\phi_n\rangle = \lim_{n\to\infty} \int_{B_R(0)} f(\phi_n-\phi_n(0)) = \int_{B_R(0)} f(\phi-\phi(0)) = \langle PV(f),\phi\rangle,$$

by the dominated convergence theorem. This shows that $PV(f)$ is continuous, hence a distribution. It agrees with f on $\mathbb{R}^n\setminus\{0\}$: if $\phi \in C_c^\infty$ is supported on $B_R(0)\setminus\{0\}$ then

$$\langle PV(f),\phi\rangle = \int_{B_R(0)} f(\phi-\phi(0)) = \int_{B_R(0)} f\phi = \int f\phi = \langle f,\phi\rangle.$$

If $\phi \in C_c^\infty$ is supported on $B_R(0)$ and $r \in (0,\infty)$ then $\phi\circ S_r^{-1} \in C_c^\infty(B_{rR}(0))$ and hence (by Theorem 2.44)

$$\langle PV(f)\circ S_r,\phi\rangle = |\det(S_r)|^{-1}\langle PV(f),\phi\circ S_r^{-1}\rangle = |\det(S_r^{-1})|\int_{B_{rR}(0)} f(\phi\circ S_r^{-1}-\phi(S_r^{-1}(0))) = \int_{B_R(0)} (f\circ S_r)(\phi-\phi(0)).$$

Clearly $f\circ S_r = r^{-n}f$, so $PV(f)\circ S_r = r^{-n}PV(f)$, which means $PV(f)$ is homogeneous of degree $-n$.