Problem 1. Write down an explicit formula for a function $u$ solving the initial-value problem

$$\begin{cases}
  u_t + b \cdot Du + cu = 0 & \text{on } \mathbb{R}^n \times (0, \infty) \\
  u = g & \text{on } \mathbb{R}^n \times \{t = 0\}
\end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Sol: Fix $x$ and $t$, and consider $z(s) := u(x + bs, t + s)$
Then

$$\dot{z}(s) = b \cdot Du + u_t$$
$$= -cu(x + bs, t + s)$$
$$= -cz(s)$$

Therefore, $z(s) = De^{-cs}$, for some constant $D$. We can solve for $D$ by letting $s = -t$. Then,

$$z(-t) = u(x - bt, 0)$$
$$= g(x - bt)$$
$$= De^t$$

i.e. $D = g(x - bt)e^{-ct}$

Thus, $u(x + bs, t + s) = g(x - bt)e^{-c(t+s)}$
and so when $s = 0$, we get $u(x, t) = g(x - bt)e^{-ct}$. \hfill \Box$

Problem 2. Prove that Laplace’s equation $\Delta u = 0$ is rotation invariant; that is, if $O$ is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$ $(x \in \mathbb{R})$
then $\Delta v = 0$.

Solution:
Let $y := Ox$, and write $O = (a_{ij})$. Thus,

$$v(x) = u(Ox)$$
$$= u(y)$$

where $y_j = \sum_{i=1}^n a_{ji}x_i$. This then gives that

$$\frac{\partial v}{\partial x_i} = \sum_{j=1}^n \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$
$$= \sum_{j=1}^n \frac{\partial u}{\partial y_j} a_{ji}$$
Thus,

\[
\begin{bmatrix}
\frac{\partial v}{\partial x_1} \\
\vdots \\
\frac{\partial v}{\partial x_n}
\end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\
\vdots & \ddots & \vdots \\
a_{1n} & \cdots & a_{nn}
\end{bmatrix} \begin{bmatrix}
\frac{\partial u}{\partial y_1} \\
\vdots \\
\frac{\partial u}{\partial y_n}
\end{bmatrix}
\]

\[
D_x \cdot v = O^T D_y \cdot u
\]

Now,

\[
\Delta v = D_x v \cdot D_x v
\]

\[
= (O^T D_x u) \cdot (O^T D_x u)
\]

\[
= (O^T D_x u)^T O^T D_x u
\]

\[
= (D_x u)^T (O^T)^T O^T D_x u
\]

\[
= (D_x u)^T O O^T D_x u
\]

\[
= (D_x u)^T D_x u \quad \text{because } O \text{ is orthogonal}
\]

\[
= (D_x u) \cdot (D_x u)
\]

\[
= \Delta u(y)
\]

\[
= 0
\]

**Problem 3.** Modify the proof of the mean value formulas to show for \( n \geq 3 \) that

\[
u(0) = \frac{1}{n \alpha(n) r^{n-1}} \int_{\partial B(0, r)} g dS + \frac{1}{n(n-2) \alpha(n)} \int_{B(0, r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,
\]

provided

\[
\begin{cases}
-\Delta u = f \quad \text{in } B^0(0, r) \\
u = g \quad \text{on } \partial B(0, r).
\end{cases}
\]

**Solution:** Set

\[
\phi(t) = \frac{1}{n \alpha(n) t^{n-1}} \int_{\partial B(0, t)} u(y) dS(y), \quad 0 \leq t < r,
\]

and

\[
\phi(r) = \frac{1}{n \alpha(n) r^{n-1}} \int_{\partial B(0, r)} u(y) dS(y) = \frac{1}{n \alpha(n) r^{n-1}} \int_{\partial B(0, r)} g dS.
\]

Then,

\[
\phi(t) = \frac{t}{n} \left( \frac{1}{\alpha(n) t^n} \int_{B(0, t)} \Delta u(y) dy \right) = \frac{t}{n} \left( \frac{1}{\alpha(n) t^n} \int_{B(0, t)} -f dy \right) = \frac{-1}{\alpha(n) t^{n-1}} \int_{B(0, t)} f dy.
\]

(See the proof of Thm2)
Let \( \epsilon > 0 \) be given.

\[
(1) \quad \phi(\epsilon) = \phi(r) - \int_{\epsilon}^{r} \phi'(t)dt = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} gds - \int_{\epsilon}^{r} \phi'(t)dt.
\]

Using integration by parts, we compute

\[
- \int_{\epsilon}^{r} \phi'(t)dt = \int_{\epsilon}^{r} \frac{1}{n\alpha(n)r^{n-1}} \int_{B(0,r)} fdydt
= \frac{1}{n\alpha(n)} \int_{\epsilon}^{r} \frac{1}{r^{n-1}} \int_{B(0,r)} fdydt
= \frac{1}{n\alpha(n)} \left( \frac{1}{2 - n} \int_{B(0,r)} fdy \right)_{\epsilon}^{r} - \frac{1}{n\alpha(n)} \int_{\epsilon}^{r} \frac{1}{2 - n} \int_{\partial B(0,r)} fds dt
= \frac{n(n-2)\alpha(n)}{n(n-2)\alpha(n)} \left( \int_{B(0,r)} fdy - \frac{1}{r^{n-2}} \int_{B(0,r)} fdy + \frac{1}{\epsilon^{n-2}} \int_{B(0,\epsilon)} fdy \right)
= \frac{1}{n(n-2)\alpha(n)} \left( I - \frac{1}{r^{n-2}} \int_{B(0,r)} fdy + J \right).
\]

Observe that

\[
J : \frac{1}{\epsilon^{n-2}} \int_{B(0,\epsilon)} fdy \leq C \cdot \epsilon^2, \quad \text{for some constant } C > 0
\]

and

\[
\int_{B(0,\epsilon)} \frac{1}{|x|^{n-2}} f(x)dx = \int_{0}^{r} dt \int_{\partial B(0,t)} \frac{1}{|x|^{n-2}} fds.
\]

As \( \epsilon \to 0 \), \( I + J \to \int_{B(0,\epsilon)} \frac{1}{|x|^{n-2}} f(x)dx \). Thus,

\[
\lim_{\epsilon \to 0} - \int_{\epsilon}^{r} \phi'(t)dt = \frac{1}{n(n-2)\alpha(n)} \left( \int_{B(0,r)} \frac{1}{|x|^{n-2}} f(x)dx - \frac{1}{r^{n-2}} \int_{B(0,r)} fdy \right)
= \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) fdx.
\]

Therefore, letting \( \epsilon \to 0 \), we have from (1)

\[
u(0) = \phi(0) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} gds + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) fdx.
\]

\[
\square
\]

**Problem 4.** We say \( v \in C^2(\tilde{U}) \) is subharmonic if

\[-\Delta v \leq 0 \quad \text{in } U.\]

(a) Prove for subharmonic \( v \) that

\[v(x) \leq \int_{B(x,r)} v \ dy \quad \text{for all } B(x,r) \subset U.\]

(b) Prove that therefore \( \max_{\partial U} v = \max_{\partial U} v \).

(c) Let \( \phi : \mathbb{R} \to \mathbb{R} \) be smooth and convex. Assume \( u \) is harmonic and \( v := \phi(u) \). Prove \( v \) is subharmonic.
(d) Prove $v := |Du|^2$ is subharmonic, whenever $u$ is harmonic.

Solution.

(a) As in the proof of Theorem 2, set $\phi(r) := \int_{\partial B(x,r)} v \, dS(y)$ and obtain

$$
\phi'(r) = \frac{r}{n} \int_{B(x,r)} \Delta v(y) dy \geq 0.
$$

For $0 < \epsilon < r$,

$$
\int_{\epsilon}^{r} \phi'(s) \, ds = \phi(r) - \phi(\epsilon) \geq 0.
$$

Hence, $\phi(r) \geq \lim_{\epsilon \to 0} \phi(\epsilon) = v(x)$. Therefore,

$$
\int_{B(x,r)} v \, dy \leq \frac{1}{\alpha(n) r^n} \int_{B(x,r)} v \, dy = \frac{1}{\alpha(n) r^n} \int_{0}^{r} \left( \int_{\partial B(x,s)} v(z) \, dS(z) \right) \, ds
$$

$$
= \frac{1}{\alpha(n) r^n} \int_{0}^{r} n(\alpha(n)) s^{n-1} \phi(s) \, ds \geq \frac{1}{r^n} \int_{0}^{r} n s^{n-1} v(x) \, ds = v(x)
$$

(b) We assume that $U \subset \mathbb{R}^n$ is open and bounded. For a moment, we assume also that $U$ is connected. Suppose that $x_0 \in U$ is such a point that $v(x_0) = M := \max_{\partial U} v$. Then for $0 < r < \text{dist}(x_0, \partial U)$,

$$
M = v(x_0) \leq \int_{B(x_0,r)} v \, dy \leq M.
$$

Due to continuity of $v$, an equality holds only if $v \equiv M$ within $B(x_0, r)$. Therefore, the set $u^{-1}(\{M\}) \cap U = \{x \in U | u(x) = M\}$ is both open and relatively closed in $U$. By the connectedness of $U$, $v$ is constant within the set $U$. Hence, it is constant within $U$ and we conclude that $\max_{\partial U} v = \max_{\partial U} v$.

Now let $\{U_i \mid i \in I\}$ be the connected components of $U$. Pick any $x \in U$ and find $j \in I$ such that $x \in U_j$. We obtain

$$
v(x) \leq \max_{\partial U_j} v = \max_{\partial U_j} v \leq \max_{\partial U} v
$$

and conclude that $\max_{\partial U} v = \max_{\partial U} v$.

(c) For $x = (x_1, \ldots, x_n) \in U$ and $1 \leq i, j \leq n$,

$$
\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = \frac{\partial^2}{\partial x_i \partial x_j} \phi(u(x)) = \phi''(u(x)) \cdot \frac{\partial u}{\partial x_i}(x) \cdot \frac{\partial u}{\partial x_j}(x) + \phi'(u(x)) \cdot \frac{\partial^2 u}{\partial x_i \partial x_j}(x).
$$

Since $\phi$ is convex, then $\phi''(x) \geq 0$ for any $x \in \mathbb{R}$. Recall that $u$ is harmonic and obtain

$$
\Delta v = \phi''(u) \cdot \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 + \Delta u = \phi''(u) \cdot \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 \geq 0.
$$

(d) We set $v := |Du|^2 = \sum_{k=1}^{n} \left( \frac{\partial u}{\partial x_k} \right)^2$. For $x = (x_1, \ldots, x_n) \in U$ and $1 \leq i, j \leq n$,

$$
\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = 2 \sum_{k=1}^{n} \left[ \frac{\partial^2 u}{\partial x_i \partial x_k}(x) \cdot \frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \frac{\partial u}{\partial x_i}(x) \cdot \frac{\partial^3 u}{\partial x_j \partial x_k \partial x_k}(x) \right].
$$
Therefore,
\[
\frac{\partial^2 v}{\partial x_i^2} = 2 \sum_{k=1}^{n} \left[ \left( \frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 + \frac{\partial u}{\partial x_k} \cdot \frac{\partial}{\partial x_k} \left( \frac{\partial^2 u}{\partial x_i^2} \right) \right],
\]
\[
\Delta v = 2 \sum_{1 \leq i, k \leq n} \left( \frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 + \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \cdot \frac{\partial}{\partial x_k} (\Delta u) = 2 \sum_{1 \leq i, k \leq n} \left( \frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 \geq 0.
\]

\[\Box\]

**Problem 5:** Prove that there exists a constant \( C \), depending only on \( n \), such that
\[
\max_{B(0,1)} |u| \leq C \left( \max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f| \right)
\]
whenever \( u \) is a smooth solution of
\[
\begin{cases}
-\Delta u = f & \text{in } B^0(0,1) \\
u = g & \text{on } \partial B(0,1).
\end{cases}
\]

**Proof:** Let \( M := \max_{B(0,1)} |f| \), then we define \( v(x) = u(x) + \frac{M}{2n} |x|^2 \) and \( w(x) = -u(x) + \frac{M}{2n} |x|^2 \). We first consider \( v(x) \). Note that
\[-\Delta v = -\Delta u - M = f - M \leq 0.
\]
So, \( v(x) \) is a subharmonic function.

From Problem 4 (b), we have
\[
\max_{B(0,1)} v(x) = \max_{B(0,1)} v(x) \leq \max_{\partial B(0,1)} |g| + \frac{M}{2n}.
\]
That is
\[
\max_{B(0,1)} u(x) \leq \max_{B(0,1)} v(x) \leq \max_{\partial B(0,1)} |g| + \frac{1}{2n} \max_{B(0,1)} |f|.
\]

Then, for \( w(x) \), we have
\[-\Delta w = \Delta u - M = -f - M \leq 0.
\]
Again, we can get
\[
\max_{B(0,1)} w(x) = \max_{B(0,1)} w(x) \leq \max_{\partial B(0,1)} |g| + \frac{M}{2n}.
\]
i.e.
\[
\max_{B(0,1)} -u(x) \leq \max_{B(0,1)} w(x) \leq \max_{\partial B(0,1)} |g| + \frac{1}{2n} \max_{B(0,1)} |f|.
\]
Combining these two together, we finally proved the problem. \[\Box\]

**Problem 6.** Use Poisson’s formula for the ball to prove
\[
r^{n-2} \frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0)
\]
whenever \( u \) is positive and harmonic in \( B^0(0, r) \). This is an explicit form of Harnack’s inequality.
Solution.
Since $y \in \partial B(0, r)$, then $|x - y| \leq |x| + r$. Therefore,

\[
\begin{align*}
  u(x) &= \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x - y|^n} dS(y) \\
  &\geq \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{(r + |x|)^n} dS(y) = \frac{r^{n-2}}{(r + |x|)^{n-1}} \cdot \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} g(y) dS(y) \\
  &= \frac{r^{n-2}}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} g(y) dS(y) = \frac{r^{n-2}}{(r + |x|)^{n-1}} u(0)
\end{align*}
\]

The inequality $u(x) \leq \frac{r^{n-2}}{(r - |x|)^{n-1}} u(0)$ can be proven in a similar way. \(\square\)

**Problem 7.** Prove Poisson’s formula for a ball: Assume $g \in C(\partial B(0, r))$ and let

\[
u(x) = \frac{r^2 - x^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x - y|^n} dS(y) \quad \text{for} \quad x \in B^n(0, r).
\]

Show that

**Proof.**

**Problem 8.**

Let $u$ be the solution of

\[
\begin{align*}
  \Delta u &= 0 \quad \text{in} \quad \mathbb{R}^n_+ \\
  u &= g \quad \text{on} \quad \partial \mathbb{R}^n_+
\end{align*}
\]

given by Poisson’s formula for the half-space. Assume $g$ is bounded and $g(x) = |x|$ for $x \in \partial \mathbb{R}^n_+$, $|x| \leq 1$. Show $Du$ is not bounded near $x = 0$. (Hint: Estimate $\frac{\mu(\lambda e_n) - u(0)}{\lambda}$.)

**Proof:** From formula (33) on page 37, we have

\[
u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x - y|^n} dy,
\]

and $u(0) = g(0) = 0$. Thus, using hint, we get

\[
\begin{align*}
  \frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{2}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{\lambda e_n - y^n} dy \\
  &= \frac{2}{n\alpha(n)} \int_{|y| \leq 1 \cap \partial \mathbb{R}^n_+} \frac{g(y)}{\lambda e_n - y^n} dy + \frac{2}{n\alpha(n)} \int_{|y| > 1 \cap \partial \mathbb{R}^n_+} \frac{g(y)}{\lambda e_n - y^n} dy
\end{align*}
\]

Taking absolute value on both sides, we have

\[
\left| \frac{u(\lambda e_n) - u(0)}{\lambda} \right| \geq \left| \frac{2}{n\alpha(n)} \int_{|y| \leq 1 \cap \partial \mathbb{R}^n_+} \frac{g(y)}{\lambda e_n - y^n} dy \right| - \left| \frac{2}{n\alpha(n)} \int_{|y| > 1 \cap \partial \mathbb{R}^n_+} \frac{|g(y)|}{\lambda e_n - y^n} dy \right|
\]

\[= I_1 - I_2.\]
Since $g$ is bounded, so it is obvious that $I_2$ is bounded and independent of $\lambda$. For $I_1$, in this case, $g(y) = |y|$, so

$$I_1 = \frac{2}{na(n)} \int_{|y| \leq 1} \frac{|y|}{|\lambda e_n - y|^n} dy \geq \frac{2}{na(n)} \int_{|y| \leq 1} \frac{|y|}{(\lambda + |y|)^n} dy$$

Note that for fixed $y$, $\frac{|y|}{(\lambda + |y|)^n}$ is increasing when $\lambda$ is decreasing to 0, so by Monotone Convergence theorem, we have

$$\lim_{\lambda \to 0} \frac{2}{na(n)} \int_{|y| \leq 1} \frac{|y|}{(\lambda + |y|)^n} dy = \int_{|y| \leq 1} \frac{|y|}{|y|^n} dy = \int_{B_{n-1}(0,1)} \frac{|y|}{|y|^n} dy = \int_{0}^{1} dr \int_{\partial B_{n-1}(0,r)} \frac{1}{|y|^{n-1}} dS(y) = C \int_{0}^{1} \frac{1}{r^{n-1}} r^{n-2} dr = \infty.$$ 

So, $Du$ is unbounded near $x = 0$. \qed

**Problem 10.**

Suppose $u$ is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(i) Show $u_{\lambda t}(x, t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(ii) Use (i) to show $v(x, t) := x \cdot Du(x, t) + 2tu_t(x, t)$ solves the heat equation as well.

(i) $u_{\lambda t}(x, t) = \lambda^2 u_t(\lambda x, \lambda^2 t)$ and $u_{\lambda x_i}(x, t) = \lambda u(\lambda x, \lambda^2 t)$ for each $i$. Then $u_{\lambda x_i x_i}(x, t) = \lambda^2 u_{x_i}(\lambda x, \lambda^2 t)$. Consequently, $\Delta u_\lambda = \lambda^2 \Delta u$ and $u_t - \Delta u_\lambda = \lambda^2 (u_t - \Delta u)$, so $u_\lambda$ solves the heat equation for all $\lambda \in \mathbb{R}$.

(ii) We differentiate $u(\lambda x, \lambda^2 t) = u(\lambda x_1, \ldots, \lambda x_n, \lambda^2 t)$ with respect to $\lambda$ we get

$$\sum_k x_k u_{x_k}(\lambda x_1, \ldots, \lambda x_k, \lambda^2 t) + 2\lambda tu_t(\lambda x_1, \ldots, \lambda x_n, \lambda^2 t) = x \cdot D(\lambda x, \lambda^2 t) + 2tu_t(\lambda x, \lambda^2 t).$$

Taking $\lambda = 1$, we then have that $v(x, t) = x \cdot Du(x, t) + 2tu_t(x, t)$. $u$ is smooth, so the second derivatives of $u(\lambda x, \lambda^2 t)$ are continuous, meaning the mixed partials are equal. Therefore, $v_t - \Delta v = \frac{\partial}{\partial \lambda} u(\lambda x, \lambda^2 t) - \Delta \frac{\partial}{\partial \lambda} u(\lambda x, \lambda^2 t) = \frac{\partial}{\partial \lambda} u(\lambda x, \lambda^2 t) - \frac{\partial}{\partial \lambda} \Delta u(\lambda x, \lambda^2 t) = \frac{\partial}{\partial \lambda} (u_t - \Delta u) = 0$, since $u_\lambda$ satisfies the heat equation for all $\lambda$. Thus $v$ does as well.

**Problem 11:** Assume $n = 1$ and $u(x, t) = v(\frac{x^2}{t})$.

a) Show

$$u_t = u_{xx}$$

if and only if

$$4zv''(z) + (2 + z)v'(z) = 0 \quad (z > 0)$$
b) Show that the general solution of (1) is

\[ v(z) = c \int_{0}^{z} e^{-s/4} s^{-1/2} ds + d \]

c) Differentiate \( v \left( \frac{x^2}{t} \right) \) with respect to \( x \) and select the constant \( c \) properly, so as to obtain the fundamental solution \( \Phi \) for \( n = 1 \).

**Solution:**

a) Assume that \( u_t = u_{xx} \). Then

\[ u_t = -\frac{x^2}{t^2} v' \left( \frac{x^2}{t} \right) \]

and

\[ u_{xx} = 2v' \left( \frac{x^2}{t} \right) + 4x^2 v'' \left( \frac{x^2}{t} \right) \]

So \( u_t = u_{xx} \) implies that

\[ -\frac{x^2}{t^2} v' \left( \frac{x^2}{t} \right) = 2v' \left( \frac{x^2}{t} \right) + 4x^2 v'' \left( \frac{x^2}{t} \right) \]

or

\[ 4x^2 \frac{v''}{t^2} \left( \frac{x^2}{t} \right) + \left( \frac{2}{t} + \frac{x^2}{t^2} \right) v' \left( \frac{x^2}{t} \right) = 0 \]

If we let \( z = \frac{x^2}{t} \), we get

\[ \frac{4z}{t} v''(z) + \left( \frac{2}{t} + \frac{z}{t} \right) v'(z) = 0 \]

Multiplying this equation by \( t \) gives the desired equality.

For the other direction, reverse the steps, and hence our proof is done.

b) \[ 4zv'' + (2 + z)v' = 0 \]

\[ \Rightarrow \]

\[ \frac{v''}{v'} = \frac{1}{2} \frac{1}{z} - \frac{1}{4} \]

\[ \Rightarrow \]

(by integrating)

\[ \log(v') = -\log \sqrt{z} - \frac{z}{4} + c \]

\[ \Rightarrow \]

\[ v' = Cz^{-1/2} e^{-z/4} \]

\[ \Rightarrow \]

\[ v = C \int_{0}^{z} e^{-s/4} s^{-1/2} ds + d \]
as is desired.

c) 
\[ v(z) = c \int_0^z e^{-s^4} s^{-1/2} ds + d \]

\[ \Rightarrow \]
\[ v\left(\frac{x^2}{t}\right) = c \int_0^{\frac{x^2}{t}} e^{-s^4} s^{-1/2} ds + d \]

\[ \Rightarrow \]
\[ v'\left(\frac{x^2}{t}\right) = \frac{2x}{t} e^{-\frac{x^2}{4t}} \left(\frac{x^2}{t}\right)^{-1/2} \]

or
\[ v'\left(\frac{x^2}{t}\right) = \frac{2c}{\sqrt{t}} e^{-\frac{x^2}{4t}} \]

Now we want to integrate over \( \mathbb{R} \) and set the integral equal to 1. Thus we get

\[ 1 = \frac{2c}{\sqrt{t}} \int_\infty^\infty e^{-\frac{x^2}{4t}} dx \]

Letting \( y = \frac{x}{\sqrt{4t}} \), we get \( dy = (4t)^{-1/2} dx \) and substituting, we get

\[ 1 = \frac{c}{\sqrt{t}} \int_\infty^\infty \sqrt{4t} e^{-y^2} dy \]

or

\[ 1 = 4c \int_\infty^\infty e^{-y^2} dy \]

Employing the identity \( \int_\infty^\infty e^{-y^2} dy = \sqrt{\pi} \) and solving for \( c \), we get

\[ c = \frac{1}{4\sqrt{\pi}} \]

Thus,

\[ \Phi(x, t) := v'\left(\frac{x^2}{t}\right) = \frac{2c}{\sqrt{t}} e^{-\frac{x^2}{4t}} = \frac{1}{2\sqrt{\pi t}} e^{\frac{x^2}{4t}} \]

is easily shown to solve the equation

\[ \Phi_t = \Phi_{xx} \]

\[ \square \]

**Problem 12.** Write down an explicit formula for a solution of

\[ \begin{cases} 
    u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty) \\
    u = g & \text{on } \mathbb{R}^n \times \{t = 0\},
\end{cases} \]

where \( c \in \mathbb{R} \).
Solution: Set \( v(x, t) = u(x, t)e^{Ct} \). Then, \( v_t = u_te^{Ct} + Ce^{Ct}u \) and \( v_{x,t} = u_{x,t}e^{Ct} \).

\[
\begin{align*}
    v_t - \Delta v &= u_te^{Ct} + Ce^{Ct}u - e^{Ct}\Delta u \\
    &= e^{Ct}(u_t - \Delta u + Cu) \\
    &= e^{Ct}f.
\end{align*}
\]

So, \( v \) is a solution of

\[
\begin{cases}
    v_t - \Delta v = e^{Ct}f & \text{in } \mathbb{R}^n \times (0, \infty) \\
    v = g & \text{on } \mathbb{R}^n \times \{t = 0\},
\end{cases}
\]

By (17) (p.51),

\[
v(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)e^{Cs}f(y, s)dyds
\]

where \( \Phi \) is the fundamental solution of the heat equation. Since \( v(x, t) = u(x, t)e^{Ct} \), we have

\[
u(x, t) = e^{Ct}\left( \int_{\mathbb{R}^n} \Phi(x - y, t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)e^{Cs}f(y, s)dyds \right).
\]

Problem 13: Given \( g : [0, \infty) \rightarrow \mathbb{R} \), with \( g(0) = 0 \), derive the formula

\[
u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{-\frac{x^2}{4(t - s)}} g(s)ds, x > 0
\]

for a solution of the initial/boundary-value problem

\[
\begin{cases}
    u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ x (0, \infty) \\
    u = 0 & \text{on } \mathbb{R}_+ x \{t = 0\}, \\
    u = g & \text{on } \{x = 0\} x [0, \infty).
\end{cases}
\]

Proof. We define

\[
v(x, t) = \begin{cases}
    u(x, t) - g(t) & x > 0, \\
    -u(-x, t) + g(t) & x \leq 0.
\end{cases}
\]

So, we have

\[
v_t(x, t) = \begin{cases}
    u_t(x, t) - g'(t) & x > 0, \\
    -u_t(-x, t) + g'(t) & x \leq 0,
\end{cases}
\]

and

\[
v_{xx}(x, t) = \begin{cases}
    u_{xx}(x, t) & x > 0, \\
    -u_{xx}(-x, t) & x \leq 0.
\end{cases}
\]
Hence,

\[
\begin{cases}
  v_i(x, t) - v_{xx}(x, t) = \begin{cases} -g'(t) & x > 0, \\ g'(t) & x \leq 0. \end{cases} \\
  v(x, 0) = 0, \\
  v(0, t) = 0.
\end{cases}
\]

By formula (13) on page 49, we get

\[
v(x, t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left\{ \int_{-\infty}^0 e^{\frac{-(y-s)^2}{4(t-s)}} g'(s)dyds - \int_0^\infty e^{\frac{-(y-s)^2}{4(t-s)}} g'(s)dyds \right\}
\]

Note that(page 46 Lemma)

\[
\int_{-\infty}^\infty \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(y-s)^2}{4(t-s)}} dy = 1,
\]

so when \(x > 0\), we let \(y - x = -z\) and obtain

\[
u(x, t) = v(x, t) + g(t) = v(x, t) + \int_0^t g'(s)ds \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(y-s)^2}{4(t-s)}} dy
\]

\[
= 2 \int_0^t \frac{1}{\sqrt{4\pi}} (t-s)^{-1/2} \int_{-\infty}^0 e^{-\frac{(y-s)^2}{4(t-s)}} dy g'(s)ds
\]

\[
= \int_0^t \frac{1}{\sqrt{\pi}} (t-s)^{-1/2} \int_x^\infty e^{\frac{z^2}{4(t-s)}} dz dg(s)
\]

Integrating by parts, we get

\[
u(x, t) = \frac{1}{\sqrt{\pi}} (t-s)^{-1/2} \int_x^\infty e^{\frac{z^2}{4(t-s)}} dz g(s)|_{s=0}
\]

\[- \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{z^2}{4(t-s)}} dz
\]

\[- \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-1/2} ds \int_x^\infty e^{\frac{z^2}{4(t-s)}} \frac{-z^2}{4(t-s)^2} dz
\]

\[= I_1 - \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{z^2}{4(t-s)}} dz
\]

\[+ \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-1/2} ds \int_x^\infty \frac{-z}{2(t-s)} de^{\frac{z^2}{4(t-s)}}
\]

\[= I_1 - \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{z^2}{4(t-s)}} dz
\]

\[- \int_0^t g(s) \frac{1}{\sqrt{4\pi}} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{z^2}{4(t-s)}} dz
\]

\[+ \int_0^t g(s) \frac{1}{\sqrt{4\pi}} (t-s)^{-1/2} ds \int_x^\infty \frac{-z}{2(t-s)} e^{\frac{z^2}{4(t-s)}}|_{z=\infty} \]

\[+ \int_0^t g(s) \frac{1}{\sqrt{4\pi}} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{z^2}{4(t-s)}} dz
\]

\[= I_1 + \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-z^2}{4(t-s)}} g(s) ds.
\]
Now, we focus on \( I_1 \) and define \( w^2 \) to be \( \frac{x^2}{4\epsilon} \).

\[
I_1 = \lim_{\epsilon \to 0^+} \frac{1}{\sqrt{\pi}} \epsilon^{-1/2} \int_x^\infty e^{-z^2/4\epsilon} \, dz \, g(t - \epsilon)
\]

\[
= g(t) \lim_{\epsilon \to 0^+} \frac{1}{\sqrt{\pi}} \int_x^\infty 2e^{-w^2} \, dw = 0.
\]

Thus, we proved

\[
u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{w^2/4(t-s)} g(s) \, ds, \quad x > 0.
\]

Next, we need to show that

\[
\lim_{x \to 0^+} \nu(x, t) = g(t).
\]

Note that for any fixed \( \delta > 0 \).

\[
\lim_{x \to 0^+} \nu(x, t) = \lim_{x \to 0^+} \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{w^2/4(t-s)} g(s) \, ds
\]

\[
+ \lim_{x \to 0^+} \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{w^2/4(t-s)} g(s) \, ds
\]

\[
= g(t) \lim_{x \to 0^+} \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{w^2/4(t-s)} \, ds
\]

\[
= g(t) \lim_{x \to 0^+} \frac{x}{\sqrt{4\pi}} \int_0^\delta \frac{1}{s^{3/2}} e^{w^2/4s} \, ds
\]

For fixed \( x \), we let \( s = x^2/w^2 \) and get

\[
\lim_{x \to 0^+} \nu(x, t) = g(t) \lim_{x \to 0^+} \frac{x}{2 \sqrt{\pi}} \int_0^\delta \frac{w^3}{x^3} e^{w^2/4x^2} - \frac{2x^2}{w^3} \, dw
\]

\[
= g(t) \lim_{x \to 0^+} \frac{1}{\sqrt{\pi}} \int_0^\infty e^{w^2/4x^2} \, dw
\]

\[
= g(t) \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-w^2/4} \, dw = g(t).
\]

Hence, we are done.

\[\square\]

**Problem 14.** We say \( v \in C_c^2(U_T) \) is a *subsolution* of the heat equation if

\[
v_t - \Delta v \leq 0 \quad \text{in} \quad U_T.
\]

(a) Prove for a subsolution \( v \) that

\[
v(x, t) \leq \frac{1}{4\pi t} \int \int_{E(x, t; r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} \, dyds
\]

for all \( E(x, t; r) \subset U_T \).

(b) Prove that therefore \( \max_{\bar{U}_T} v = \max_{\Gamma_T} v \)
Solution.

(a) We may well assume upon translating the space and time coordinates that $x = 0$ and $t = 0$. As in the proof of Theorem 3, set

$$
\phi(r) := \frac{1}{r^n} \int \int_{E(r)} v(y, s) \frac{|y|^2}{s^2} dy ds,
$$

$$
\psi(y, s) := -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r
$$

and derive

$$
\phi'(r) \geq \frac{1}{r^{n+1}} \int \int_{E(r)} -4n\Delta \psi - \frac{2n}{s} \sum_{i=1}^{n} v_{y_i} y_i dy ds
$$

$$
= \sum_{i=1}^{n} \frac{1}{r^{n+1}} \int \int_{E(r)} 4nv_{y_i} \psi_{y_i} - \frac{2n}{s} v_{y_i} y_i dy ds = 0.
$$

For $0 < \epsilon < r$,

$$
\int_{\epsilon}^{r} \phi'(z) dz = \phi(r) - \phi(\epsilon) \geq 0.
$$

Hence, $\phi(r) \geq \lim_{\epsilon \to 0} \phi(\epsilon) = v(0, 0) \cdot \lim_{\epsilon \to 0} \frac{1}{r^{n+1}} \int \int_{E(\epsilon)} \frac{|y|^2}{s^2} dy ds = 4v(0, 0)$, and the statement follows.

(b) Suppose there exists a point $(x_0, t_0) \in U_T$ with $u(x_0, t_0) = M := \max_{E(x_0, t_0)} u$. Then for all sufficiently small $r > 0$, $E(x_0, t_0; r) \subset U_T$. Using the result proved above, we deduce

$$
M = v(x_0, t_0) \leq \frac{1}{4r^n} \int \int_{E(x_0, t_0; r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds \leq M,
$$

since

$$
1 = \frac{1}{4r^n} \int \int_{E(x_0, t_0; r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} dy ds.
$$

Conclude that $u|_{E(x_0, t_0; r)} = M$. The argument used in the proof of Theorem 4 will finish the proof.

\[\Box\]

Problem 15.

(a) Show the general solution of the PDE $u_{xy} = 0$ is

$$
u(x, y) = F(x) + G(y)
$$

for arbitrary functions $F, G$.

(b) Using the change of variables $\xi = x + t, \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Use (a), (b) to rederive d’Alembert’s formula.

Solution:

(a)

$u_{xy} = 0 \Rightarrow u_x = f(x) \Rightarrow u(x, y) = \int f(x) dx + G(y)$

$u_{yx} = 0 \Rightarrow u_y = g(y) \Rightarrow u(x, y) = \int g(y) dy + F(x)$
This implies \( u(x, y) = F(x) + G(y) \).

(b) \( x = \frac{\xi + \eta}{2}, y = \frac{\xi - \eta}{2} \)

Define \( \tilde{u} := u(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}) \)

\[
\tilde{u}_\xi = \frac{1}{2}u_x + \frac{1}{2}u_t \quad \text{and} \quad \tilde{u}_\eta = \frac{1}{4}u_{xx} - \frac{1}{4}u_{tt} + \frac{1}{4}u_{tx} - \frac{1}{4}u_{tt} = \frac{1}{4}(u_{xx} - u_{tt})
\]

Hence, \( \tilde{u}_{\xi\eta} = 0 \Leftrightarrow u_{tt} - u_{xx} = 0 \).

(c) By (b), \( u_{tt} - u_{xx} = 0 \Rightarrow u_{\xi\eta} = 0 \), and \( u(\xi, \eta) = F(\xi) + G(\eta) \) by (a)

,i.e, \( u(x, y) = F(x + t) + G(x - t) \).

Since \( u(x, 0) = g, u_t(x, 0) = h \),

\[
\begin{align*}
(3) & \quad u(x, 0) = F(x) + G(x) = g(x), \\
& \quad u_t(x, 0) = F'(x) - G'(x) = h(x)
\end{align*}
\]

Integration \( \Rightarrow \)

\[
(4) \quad F(x) - G(x) = \int_0^x h(y)dy + C, \quad C: \text{constant.}
\]

\[
(2) + (3); \quad F(x) = \frac{1}{2}(g(x) + \int_0^x h(y)dy + C)
\]

\[
(2) - (3); \quad G(x) = \frac{1}{2}(g(x) - \int_0^x h(y)dy - C)
\]

Thus,

\[
\begin{align*}
 u(x, y) = F(x + t) + G(x - t) & = \frac{1}{2}(g(x + t) + \int_0^{x+t} h(y)dy + C) + \frac{1}{2}(g(x - t) - \int_0^{x-t} h(y)dy - C) \\
 & = \frac{1}{2}(g(x + t) + \int_0^{x+t} h(y)dy + C + g(x - t) + \int_0^{x-t} h(y)dy - C) \\
 & = \frac{1}{2}[g(x + t) + g(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y)dy \quad (x \in \mathbb{R}, t \geq 0).
\end{align*}
\]

□

Problem 16.

Assume \( \mathbf{E} = (E^1, E^2, E^3) \) and \( \mathbf{B} = (B^1, B^2, B^3) \) solve Maxwell’s equations:

\[
\begin{align*}
\mathbf{E}_t & = \text{curl} \mathbf{B} \\
\mathbf{B}_t & = -\text{curl} \mathbf{E} \\
\text{div} \mathbf{B} & = \text{div} \mathbf{E} = 0
\end{align*}
\]

Show that \( u_{tt} - \Delta u = 0 \) where \( u = B^i \) or \( E^i \) for \( i = 1, 2, 3 \).

Solution.
curl(curl $E$) = curl($-B_i$)

$$= \left( -\frac{\partial^2 B^3}{\partial y \partial t} + \frac{\partial^2 B^2}{\partial z \partial t}, -\frac{\partial^2 B^3}{\partial x \partial t} + \frac{\partial^2 B^1}{\partial z \partial t}, -\frac{\partial^2 B^2}{\partial x \partial t} + \frac{\partial B^1}{\partial y \partial t} \right)$$

$$= \frac{\partial}{\partial t}\text{curl}\ B$$

$$= \frac{\partial}{\partial t}E_t$$

$$= \frac{\partial^2 E}{\partial t^2}$$

However, we also know that curl(curl $E$) = $\nabla(\text{div} E) - \nabla^2 E = -\nabla^2 E$. Then $E^i$ satisfies $u_{tt} - \Delta u = 0$ for $i = 1, 2, 3$.

Similarly, curl(curl $B$) = curl $E_t = -\frac{\partial^2 B}{\partial t^2}$, and curl(curl $B$) = $\nabla(\text{div} B) - \nabla^2 B = -\nabla^2 B$, so $B^i$ satisfies $u_{tt} - \Delta u = 0$ for $i = 1, 2, 3$.

**Problem 17.** (Equipartition of energy) Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial value problem for the wave equation in one dimension:

$$\begin{align*}
\begin{cases}
u_{tt} - u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (0, \infty) \\
u = g; \quad u_t = h & \text{on} \quad \mathbb{R} \times \{t = 0\}.
\end{cases}
\end{align*}$$

Suppose $g, h$ have compact support. The kinetic energy is $k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t)dx$ and the potential energy is $p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t)dx$. Prove

(i) $k(t) + p(t)$ is constant in $t$.

(ii) $k(t) = p(t)$ for all large enough times $t$.

**Proof.** (i.) We define $e(t) = k(t) + p(t) = \frac{1}{2} \int_{-\infty}^{\infty} (u_t^2 + u_x^2) dx$. Since $g, h$ have compact support, so we have

$$\frac{de(t)}{dt} = \frac{1}{2} \int_{-\infty}^{\infty} 2u_tu_{tt} + 2u_xu_{xt} dx$$

$$= \int_{-\infty}^{\infty} u_t u_{tt} dx - \int_{-\infty}^{\infty} u_x u_t dx$$

$$= \int_{-\infty}^{\infty} u_t (u_{tt} - u_{xx}) dx = 0.$$

Hence, $e(t) \equiv e(0)$.

(ii.) By d’Alembert’s formula on page 68, we have

$$u(x, t) = \frac{1}{2} [g(x + t) + g(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y)dy.$$ 

So,

$$u_t = \frac{1}{2} [g'(x + t) - g'(x - t)] + \frac{1}{2} [h(x + t) + h(x - t)],$$

and

$$u_x = \frac{1}{2} [g'(x + t) + g'(x - t)] + \frac{1}{2} [h(x + t) - h(x - t)].$$
We assume that there exists a positive constant $M$ so that $[-M, M] \supseteq \text{supp}(g')$ and $[-M, M] \supseteq \text{supp}(h)$.

Note that for a fixed $t > M$, $-M \leq x - t \leq M \Rightarrow 0 < t - M \leq x \leq t + M$ and $-M \leq x + t \leq M \Rightarrow -t - M \leq x \leq -t + M < 0$.

Thus, when $t > M$:

(a) $0 < t - M \leq x \leq t + M$.
Then we have
$$h(x + t) = g(x + t) = 0.$$ 
So,
$$u_t^2 = \frac{1}{4} g'(x - t)^2 + \frac{1}{4} h(x - t)^2 - \frac{1}{2} g'(x - t)h(x - t) = u_x^2.$$ 
(b) $-t - M \leq x \leq -t + M < 0$.
Then,
$$u_t^2 = \frac{1}{4} g'(x + t)^2 + \frac{1}{4} h(x + t)^2 - \frac{1}{2} g'(x + t)h(x + t) = u_x^2.$$ 
(c) Otherwise
$$g'(x + t) = g'(x - t) = h(x + t) = h(x - t) = 0.$$ 
So, combining all the cases, it is obvious that when $t > M$, $k(t) = p(t)$. 

**Problem 18.** Let $u$ solve
$$\begin{cases} 
u_t - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$
where $g, h$ are smooth and have compact support. Show there exists a constant $C$ such that
$$|u(x, t)| \leq C/t \quad (x \in \mathbb{R}^3, t > 0).$$

**Solution.**

From the conditions it follows that there exist $R, M > 0$ such that $\text{spt } g, \text{spt } h \subset B(0, R)$ and $g(y) \leq M, |Dg(y)| \leq M, h(y) \leq M$ for any $y \in \mathbb{R}^3$. Kirchhoff’s formula gives the solution of the initial-value problem:
$$u(x, t) = \int_{\partial B(x, t)} th(y) + g(y) + Dg(y) \cdot (y - x) \, dS(y).$$
Denote by $\Sigma$ the intersection $\partial B(x, t) \cap B(0, R)$. Observe that the area of $\Sigma$ is not greater than the area of the sphere $\partial B(0, R)$. Then, for $t > 0$, we obtain
$$\left| \int_{\partial B(x, t)} th(y) + Dg(y) \cdot (y - x) \, dS(y) \right| = \frac{1}{4\pi t^2} \int_{\partial B(x, t) \cap B(0, R)} th(y) + Dg(y) \cdot (y - x) \, dS(y) \right| \leq \frac{1}{4\pi t^2} \int_{\partial B(x, t) \cap B(0, R)} t \cdot |h(y)| + |Dg(y)| \cdot |y - x| \, dS(y) \right| \leq \frac{1}{4\pi t^2} \cdot 4\pi R^2 \cdot (tM + tM) = \frac{2R^2M}{t}. $$
For $t > 1$, using the same argument, we get
\[
\left| \int_{\partial B(x,t)} g(y) \, dS(y) \right| = \frac{1}{4\pi t^2} \left| \int_{\partial B(x,t) \cap B(0,R)} g(y) \, dS(y) \right| \leq \frac{1}{4\pi t^2} \cdot 4\pi R^2 \cdot M = \frac{R^2 M}{t^2} \leq \frac{R^2 M}{t}.
\]

Notice now that the area $\Sigma$ is not greater than the area of the sphere $\partial B(x,t)$. Then for $0 < t \leq 1$,
\[
\left| \int_{\partial B(x,t)} g(y) \, dS(y) \right| = \frac{1}{4\pi t^2} \left| \int_{\partial B(x,t) \cap B(0,R)} g(y) \, dS(y) \right| \leq \frac{1}{4\pi t^2} \cdot 4\pi R^2 \cdot M \leq \frac{M}{t}.
\]

Without loss of generality, we can take $R > 1$. Then, combining the estimates obtained above, we conclude $|u(x,t)| \leq \frac{3R^2 M}{t}$.

\[\square\]

Evans PDE Solutions, Chapter 5

Alex: 4, Helen: 5, Rob H.: 1

Problem 1.

Suppose $k \in \{0, 1, \ldots\}$, $0 < \gamma < 1$. Prove $C^{k,\gamma}(\bar{U})$ is a Banach space.

Solution:

1. First we show that $\| \cdot \|_{C^{k,\gamma}(\bar{U})}$ is a norm, where we recall that

\[
\|u\|_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})},
\]

and

\[
[u]_{C^{0,\gamma}(\bar{U})} = \sup_{x \neq y \in U} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\}.
\]

For the sake of opaqueness we now omit subscripts on all norms unless it is unclear from context.

2. For any $\lambda \in \mathbb{R}$ we have first

\[
[u] = \sup_{x,y \in U} \left| \frac{\lambda u(x) - \lambda u(y)}{|x - y|^\gamma} \right| = |\lambda| \sup_{x,y \in U} \frac{|u(x) - u(y)|}{|x - y|^\gamma} = |\lambda| \|u\|,
\]

and certainly

\[
\|D^\alpha (\lambda u)\|_{C(\bar{U})} = |\lambda| \|D^\alpha u\| .
\]

So

\[
\|\lambda u\| = \sum_{|\alpha| \leq k} \|D^\alpha (\lambda u)\| + \sum_{|\alpha| = k} [D^\alpha (\lambda u)]
\]

\[= |\lambda| \sum_{|\alpha| \leq k} \|D^\alpha u\| + |\lambda| \sum_{|\alpha| = k} [D^\alpha u]
\]

\[= |\lambda| \cdot \|u\| .
\]

3. If $u = 0$ it is obvious that $\|u\| = 0$. On the other hand, $\|u\| = 0$ implies that

\[
\|D^\alpha u\|_{C(\bar{U})} = 0
\]
for every \(|\alpha| \leq k\). In particular this is true for \(\alpha = 0\) so that the supremum of \(D^0 u = u\) on \(U\) is 0, i.e. \(u \equiv 0\).

4. Finally we must prove the triangle inequality. We know the triangle inequality is true for the sup norm \(\| \cdot \|_{C(U)}\). We can also see that for any \(\alpha\) which makes sense
\[
[D^\alpha (u + v)] = [D^\alpha u + D^\alpha v] \leq [D^\alpha u] + [D^\alpha v].
\]
Therefore we can easily conclude
\[
\|u + v\| = \sum_{|\alpha| \leq k} \|D^\alpha (u + v)\| + \sum_{|\alpha| = k} [D^\alpha (u + v)]
\leq \sum_{|\alpha| \leq k} (\|D^\alpha u\| + \|D^\alpha v\|) + \sum_{|\alpha| = k} (\|D^\alpha u\| + \|D^\alpha v\|)
= \|u\| + \|v\|.
\]

5. We need only show that \(C^{k,\gamma}(U)\) is complete. So let \(\{u_m\}\) be a Cauchy sequence. Then \(\{u_m(x)\}\) is a Cauchy sequence for every \(x\), so define \(u\) to be the pointwise limit of the \(u_m\). Now if \(V\) is any bounded subset of \(U\), then \(\hat{V}\) is compact, so that \(u_m \Rightarrow u\) uniformly on any \(V\). Since the \(u_m\) are uniformly continuous on \(\hat{V}\) by assumption, this implies that \(u\) is uniformly continuous on \(\hat{V}\) as well (and so, a fortiori \(u \in C(U)\)). Therefore \(u \in C(\hat{U})\).

What we would really like would be to have \(u \in C^k(\hat{U})\). But similar arguments show that \(u\) has derivatives \(D^\alpha u\) for all \(|\alpha| \leq k\) on \(U\) by restricting first to bounded subsets of \(U\) to find the derivatives and then using uniform convergence on these subsets to show the derivatives must also be uniformly continuous on bounded subsets since the \(D^\alpha u_m\) were.

This leaves us with only showing that the norm of \(u\) is finite, so that in fact \(u \in C^{k,\gamma}(U)\). But for every \(n\) we have
\[
\|u_n - u\| = \sum_{|\alpha| \leq k} \sup_{x \in U} |D^\alpha u_n(x) - D^\alpha u(x)| + \sum_{|\alpha| = k} \sup_{x, y \in U} \frac{|D^\alpha u_n(x) - D^\alpha u_n(y) - D^\alpha u(x) + D^\alpha u(y)|}{|x - y|^{\gamma}}
\leq \lim_{m \to \infty} \left( \sum_{|\alpha| \leq k} \sup_{x \in U} |D^\alpha u_n(x) - D^\alpha u_m(x)| + \sum_{|\alpha| = k} \sup_{x, y \in U} \frac{|D^\alpha u_n(x) - D^\alpha u_n(y) - D^\alpha u_m(x) + D^\alpha u_m(y)|}{|x - y|^{\gamma}} \right)
\leq \lim_{m \to \infty} \|u_n - u_m\|.
\]
In particular, since \(\{u_m\}\) is Cauchy there is some \(N\) so that \(n, m \geq N\) implies \(\|u_n - u_m\| \leq 1\). Letting \(m\) approach \(\infty\), this implies that \(\|u_N - u\| < 1\). Now the triangle inequality applies to give
\[
\|u\| \leq \|u_N - u\| + \|u_N\| < 1 + \|u_N\| < \infty.
\]

\(\square\)

**Problem 4.**

Assume \(U\) is bounded and \(U \subset \bigcup_{i=1}^{N} V_i\). Show there exist \(C^\infty\) functions \(\zeta_i\) \((i = 1, \ldots, N)\) such that
\[
\begin{align*}
0 \leq \zeta_1 \leq 1, \supp \zeta_i & \subset V_i & (i = 1, \ldots, N) \\
\sum_{i=1}^{N} \zeta_i = 1 & \text{ on } U.
\end{align*}
\]

The functions \(\{\zeta_i\}_{i=1}^{N}\) for a partition of unity.
Solution. Assume $U$ is bounded and $U \subset \bigcup_{i=1}^{N} V_i$. Without loss of generality, we may assume that the $V_i$ are open, for if they are not, we can replace $V_i$ by its interior. We note that, since $U$ is bounded, $\overline{U}$ is compact. Each $x \in U$ has a compact neighbourhood $N_x$ contained in $V_i$ for some $i$. Then $\{N_x^i\}$ is an open cover of $\overline{U}$, which then has a finite subcover $N_{x_1}^1, \ldots, N_{x_N}^n$. We now let $F_i$ be the union of the $N_x$ contained in $V_i$. $F_i$ is the compact since it is the finite union of compact sets. The $C^{\infty}$ version of Urysohn’s Lemma (Folland, p.245) allows us to find smooth functions $\xi_1, \ldots, \xi_N$ such that $\xi_i = 1$ on $F_i$ and $\text{supp}(\xi_i) \subset V_i$. Since the $F_i$ cover $U$, $U \subset \{x : \sum_1^n \xi_i(x) > 0\}$. Now, we let $\xi_N = 1 - \zeta$, so $\sum_{i=1}^{N+1} \xi_i > 0$ everywhere. We then take

$$\zeta_i = \frac{\xi_i}{\sum_{j=1}^{N+1} \xi_j}$$

as our partition of unity.

**Problem 5 (Helen)** Prove that if $n = 1$ and $u \in W^{1,p}(0, 1)$ for some $1 \leq p < \infty$, then $u$ is equal a.e. to an absolutely continuous function, and $u'$ which exists a.e. belongs to $L^p(0, 1)$.

**Proof.** Since $u \in W^{1,p}(0, 1)$, so by definition on page 242 and 244, we have some function $v \in L^p(0, 1)$ such that

$$\int_{(0,1)} u D\phi dx = -\int_{(0,1)} v\phi dx, \quad \forall \phi \in C_{c}^{\infty}((0,1)).$$

Note that $v \in L^p(0, 1)$, so by Hölder’s inequality, we have $\|v\|_{L^1} \leq \|v\|_{L^p} \|1\|_{L^{1/p}} < \infty$, which means $v \in L^1(0, 1)$. Thus, we can define function $f(x)$ on $(0, 1)$ by the following formula

$$f(x) = u\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{x} v(t)dt, \quad \forall x \in (0, 1).$$

According to the Fundamental Theorem of Calculus, $f$ is absolutely continuous. Now we will prove $u = f$ a.e.

By the definition of $f$, we have $f' = v$ a.e. So for any $\phi \in C_{c}^{\infty}((0,1))$ we get

$$\int_{(0,1)} f D\phi dx = -\int_{(0,1)} f'\phi dx = -\int_{(0,1)} v\phi dx.$$

Therefore,

$$\int_{(0,1)} (f - u) D\phi dx = 0 \quad \forall \phi \in C_{c}^{\infty}((0,1)),$$

which means $u = f + \text{const.}$ And note that $u\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right)$, hence $u = f$ a.e. So $u'$ exists a.e. and satisfy $u' = v$ a.e., so $u' \in L^p(0, 1)$. □