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Evans PDE Solutions, Chapter 2

Joe: 1, 2,11; Denis: 4, 6, 14, 18; Minsu: 2,3, 15; Helen: 5,8,13,17. Alex:10, 16

Problem 1. Write down an explicit formula for a function u solving the initial-value problem

$$\begin{cases} u_t + b \cdot Du + cu &= 0 \quad \text{on } \mathbb{R}^n \times (0, \infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Sol: Fix x and t, and consider $z(s) := u(x + bs, t + s)$
Then

$$\begin{aligned} \dot{z}(s) &= b \cdot Du + u_t \\ &= -cu(x + bs, t + s) \\ &= -cz(s) \end{aligned}$$

Therefore, $z(s) = De^{-cs}$, for some constant D. We can solve for D by letting $s = -t$. Then,

$$\begin{aligned} z(-t) &= u(x - bt, 0) \\ &= g(x - bt) \\ &= De^{ct} \end{aligned}$$

i.e. $D = g(x - bt)e^{-ct}$
Thus, $u(x + bs, t + s) = g(x - bt)e^{-c(t+s)}$
and so when $s = 0$, we get $\boxed{u(x, t) = g(x - bt)e^{-ct}}$. □

Problem 2. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ox) \quad (x \in \mathbb{R})$$

then $\Delta v = 0$.

Solution:
Let $y := Ox$, and write $O = (a_{ij})$. Thus,

$$\begin{aligned} v(x) &= u(Ox) \\ &= u(y) \end{aligned}$$

where $y_j = \sum_{i=1}^{n} a_{ji} x_i$. This then gives that

$$\begin{aligned} \frac{\partial v}{\partial x_i} &= \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} \\ &= \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} a_{ji} \end{aligned}$$

Thus,

$$\begin{bmatrix} \frac{\partial v}{\partial x_1} \\ \vdots \\ \frac{\partial v}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial y_1} \\ \vdots \\ \frac{\partial u}{\partial y_n} \end{bmatrix}$$

$$= O^T \begin{bmatrix} \frac{\partial u}{\partial y_1} \\ \vdots \\ \frac{\partial u}{\partial y_n} \end{bmatrix}$$

$$D_x \cdot v = O^T D_y \cdot u$$

Now,

$$\begin{aligned} \Delta v &= D_x v \cdot D_x v \\ &= (O^T D_y u) \cdot (O^T D_y u) \\ &= (O^T D_y u)^T O^T D_y u \\ &= (D_y u)^T (O^T)^T O^T D_y u \\ &= (D_y u)^T O O^T D_y u \\ &= (D_y u)^T D_y u \qquad \text{because } O \text{ is orthogonal} \\ &= (D_y u) \cdot (D_y u) \\ &= \Delta u(y) \\ &= 0 \end{aligned}$$

Problem 3. Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \Big(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\Big) f dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0, r) \\ \quad u = g & \text{on } \partial B(0, r). \end{cases}$$

Solution: Set

$$\phi(t) = \frac{1}{n\alpha(n)t^{n-1}} \int_{\partial B(0,t)} u(y) dS(y), \quad 0 \leq t < r,$$

and

$$\phi(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} u(y) dS(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} g dS.$$

Then,

$$\phi\prime(t) = \frac{t}{n}\Big(\frac{1}{\alpha(n)t^n} \int_{B(0,t)} \Delta u(y) dy\Big) = \frac{t}{n}\Big(\frac{1}{\alpha(n)t^n} \int_{B(0,t)} -f dy\Big) = \frac{-1}{\alpha(n)t^{n-1}} \int_{B(0,t)} f dy.$$

(See the proof of Thm2)

Let $\epsilon > 0$ be given.

$$\phi(\epsilon) = \phi(r) - \int_\epsilon^r \phi\prime(t)dt = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} g dS - \int_\epsilon^r \phi\prime(t)dt. \tag{1}$$

Using integration by parts, we compute

$$\begin{aligned}
-\int_\epsilon^r \phi\prime(t)dt &= \int_\epsilon^r \frac{1}{n\alpha(n)t^{n-1}} \int_{B(0,t)} f dy dt \\
&= \frac{1}{n\alpha(n)} \int_\epsilon^r \frac{1}{t^{n-1}} \int_{B(0,t)} f dy dt \\
&= \frac{1}{n\alpha(n)} \Big(\Big[\frac{1}{2-n} \frac{1}{t^{n-2}} \int_{B(0,t)} f dy \Big]_\epsilon^r - \int_\epsilon^r \frac{1}{2-n} \frac{1}{t^{n-2}} \int_{\partial B(0,t)} f dS\, dt \Big) \\
&= \frac{1}{n(n-2)\alpha(n)} \Big(\int_\epsilon^r \frac{1}{t^{n-2}} \int_{\partial B(0,t)} f dS\, dt - \frac{1}{r^{n-2}} \int_{B(0,r)} f dy + \frac{1}{\epsilon^{n-2}} \int_{B(0,\epsilon)} f dy \Big) \\
&=: \frac{1}{n(n-2)\alpha(n)} \Big(I - \frac{1}{r^{n-2}} \int_{B(0,r)} f dy + J \Big).
\end{aligned}$$

Observe that

$$J : \frac{1}{\epsilon^{n-2}} \int_{B(0,\epsilon)} f dy \le C \cdot \epsilon^2, \quad \text{for some constant } C > 0$$

and

$$\int_{B(0,\epsilon)} \frac{1}{|x|^{n-2}} f(x) dx = \int_0^r dt \int_{\partial B(0,t)} \frac{1}{t^{n-2}} f dS.$$

As $\epsilon \to 0$, $I + J \to \int_{B(0,\epsilon)} \frac{1}{|x|^{n-2}} f(x)dx$. Thus,

$$\begin{aligned}
\lim_{\epsilon \to 0} - \int_\epsilon^r \phi\prime(t)dt &= \frac{1}{n(n-2)\alpha(n)} \Big(\int_{B(0,r)} \frac{1}{|x|^{n-2}} f(x)dx - \frac{1}{r^{n-2}} \int_{B(0,r)} f dy \Big) \\
&= \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \Big(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \Big) f dx.
\end{aligned}$$

Therefore, letting $\epsilon \to 0$, we have from (1)

$$u(0) = \phi(0) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \Big(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \Big) f dx.$$

□

Problem 4. We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \le 0 \quad \text{in } U.$$

(a) Prove for subharmonic v that

$$v(x) \le ⨍_{B(x,r)} v \, dy \quad \text{for all } B(x,r) \subset U.$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Solution.

(a) As in the proof of Theorem 2, set $\phi(r) := \fint_{\partial B(x,r)} v\, dS(y)$ and obtain

$$\phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta v(y) dy \geq 0.$$

For $0 < \epsilon < r$,

$$\int_\epsilon^r \phi'(s) ds = \phi(r) - \phi(\epsilon) \geq 0.$$

Hence, $\phi(r) \geq \lim_{\epsilon \to 0} \phi(\epsilon) = v(x)$. Therefore,

$$\begin{aligned} \fint_{B(x,r)} v\, dy &= \frac{1}{\alpha(n) r^n} \int_{B(x,r)} v\, dy = \frac{1}{\alpha(n) r^n} \int_0^r \left(\int_{\partial B(x,s)} v(z)\, dS(z) \right) ds \\ &= \frac{1}{\alpha(n) r^n} \int_0^r n\alpha(n) s^{n-1} \phi(s)\, ds \geq \frac{1}{r^n} \int_0^r n s^{n-1} v(x)\, ds = v(x) \end{aligned}$$

(b) We assume that $U \subset \mathbb{R}^n$ is open and bounded. For a moment, we assume also that U is connected. Suppose that $x_0 \in U$ is such a point that $v(x_0) = M := \max_{\bar{U}} v$. Then for $0 < r < \text{dist}(x_0, \partial U)$,

$$M = v(x_0) \leq \fint_{B(x_0,r)} v\, dy \leq M.$$

Due to continuity of v, an equality holds only if $v \equiv M$ within $B(x_0, r)$. Therefore, the set $u^{-1}(\{M\}) \cap U = \{x \in U | u(x) = M\}$ is both open and relatively closed in U. By the connectedness of U, v is constant within the set U. Hence, it is constant within $\bar{U}$ and we conclude that $\max_{\bar{U}} v = \max_{\partial U} v$.

Now let $\{U_i | i \in I\}$ be the connected components of U. Pick any $x \in U$ and find $j \in I$ such that $x \in U_j$. We obtain

$$v(x) \leq \max_{\bar{U}_j} v = \max_{\partial U_j} v \leq \max_{\partial U} v$$

and conclude that $\max_{\bar{U}} v = \max_{\partial U} v$.

(c) For $x = (x_1, ..., x_n) \in U$ and $1 \leq i, j \leq n$,

$$\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = \frac{\partial^2}{\partial x_i \partial x_j} \phi(u(x)) = \phi''(u(x)) \cdot \frac{\partial u}{\partial x_i}(x) \cdot \frac{\partial u}{\partial x_j}(x) + \phi'(u(x)) \cdot \frac{\partial^2 u}{\partial x_i \partial x_j}(x).$$

Since ϕ is convex, then $\phi''(x) \geq 0$ for any $x \in \mathbb{R}$. Recall that u is harmonic and obtain

$$\Delta v = \phi''(u) \cdot \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 + \Delta u = \phi''(u) \cdot \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 \geq 0.$$

(d) We set $v := |Du|^2 = \sum_{k=1}^n \left(\frac{\partial u}{\partial x_k} \right)^2$. For $x = (x_1, ..., x_n) \in U$ and $1 \leq i, j \leq n$,

$$\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = 2 \sum_{k=1}^n \left[\frac{\partial^2 u}{\partial x_i \partial x_k}(x) \cdot \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \frac{\partial u}{\partial x_k}(x) \cdot \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}(x) \right].$$

Therefore,

$$\frac{\partial^2 v}{\partial {x_i}^2} = 2\sum_{k=1}^{n}\left[\left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right)^2 + \frac{\partial u}{\partial x_k}\cdot\frac{\partial}{\partial x_k}\left(\frac{\partial^2 u}{\partial {x_i}^2}\right)\right],$$

$$\Delta v = 2\sum_{1\le i,k\le n}\left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right)^2 + \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\cdot\frac{\partial}{\partial x_k}(\Delta u) = 2\sum_{1\le i,k\le n}\left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right)^2 \ge 0.$$

□

Problem 5: Prove that there exists a constant C, depending only on n, such that

$$\max_{B(0,1)}|u| \le C\left(\max_{\partial B(0,1)}|g| + \max_{B(0,1)}|f|\right)$$

whenever u is a smooth solution of

$$\begin{cases} -\triangle u = f & \text{in} \quad B^0(0,1) \\ u = g & \text{on} \quad \partial B(0,1). \end{cases}$$

Proof: Let $M := \max_{B(0,1)}|f|$, then we define $v(x) = u(x) + \frac{M}{2n}|x|^2$ and $w(x) = -u(x) + \frac{M}{2n}|x|^2$. We first consider $v(x)$. Note that

$$-\triangle v = -\triangle u - M = f - M \le 0.$$

So, $v(x)$ is a subharmonic funcion.
From Problem 4 (b), we have

$$\max_{B(0,1)} v(x) = \max_{\partial B(0,1)} v(x) \le \max_{\partial B(0,1)}|g| + \frac{M}{2n}.$$

That is

$$\max_{B(0,1)} u(x) \le \max_{B(0,1)} v(x) \le \max_{\partial B(0,1)}|g| + \frac{1}{2n}\max_{B(0,1)}|f|.$$

Then, for $w(x)$, we have

$$-\triangle w = \triangle u - M = -f - M \le 0.$$

Again, we can get

$$\max_{B(0,1)} w(x) = \max_{\partial B(0,1)} w(x) \le \max_{\partial B(0,1)}|g| + \frac{M}{2n}.$$

i.e.

$$\max_{B(0,1)} -u(x) \le \max_{B(0,1)} w(x) \le \max_{\partial B(0,1)}|g| + \frac{1}{2n}\max_{B(0,1)}|f|.$$

Combining these two together, we finally proved the problem. □

Problem 6. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Solution.
Since $y \in \partial B(0,r)$, then $|x-y| \leq |x| + r$. Therefore,

$$
\begin{aligned}
u(x) &= \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y) \\
&\geq \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{(r+|x|)^n}dS(y) = r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}\cdot\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(0,r)}g(y)dS(y) \\
&= r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}⨍_{\partial B(0,r)}g(y)dS(y) = r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0)
\end{aligned}
$$

The inequality $u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$ can be proven in a similar way. □

Problem 7. Prove *Poisson's formula for a ball*: Assume $g \in C(\partial B(0,r))$ and let

$$
u(x) = \frac{r^2-x^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y) \text{ for } x \in B^0(0,r).
$$

Show that

Proof.

Problem 8.

Let u be the solution of

$$
\begin{cases}
\triangle u = 0 & \text{in} \quad \mathbb{R}^n_+ \\
u = g & \text{on} \quad \partial\mathbb{R}^n_+
\end{cases}
$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x|\ le 1$. Show Du is not bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n)-u(0)}{\lambda}$.)

Proof: From formula (33) on page 37, we have

$$
u(x) = \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|x-y|^n}dy,
$$

and $u(0) = g(0) = 0$. Thus, using hint, we get

$$
\begin{aligned}
\frac{u(\lambda e_n)-u(0)}{\lambda} &= \frac{2}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|\lambda e_n - y|^n}dy \\
&= \frac{2}{n\alpha(n)}\int_{|y|\leq 1 \bigcap \partial\mathbb{R}^n_+}\frac{g(y)}{|\lambda e_n - y|^n}dy + \frac{2}{n\alpha(n)}\int_{|y|>1 \bigcap \partial\mathbb{R}^n_+}\frac{g(y)}{|\lambda e_n - y|^n}dy
\end{aligned}
$$

Taking absolute value on both sides, we have

$$
\begin{aligned}
\Big|\frac{u(\lambda e_n)-u(0)}{\lambda}\Big| \geq \quad & \Big|\frac{2}{n\alpha(n)}\int_{|y|\leq 1 \bigcap \partial\mathbb{R}^n_+}\frac{g(y)}{|\lambda e_n - y|^n}dy\Big| - \frac{2}{n\alpha(n)}\int_{|y|>1 \bigcap \partial\mathbb{R}^n_+}\frac{|g(y)|}{|\lambda e_n - y|^n}dy \\
=& \mathbf{I}_1 - \mathbf{I}_2.
\end{aligned}
$$

Since g is bounded, so it is obvious that $\mathbf{I}_2$ is bounded and independent of λ. For $\mathbf{I}_1$, in this case, $g(y) = |y|$, so

$$\begin{aligned}\mathbf{I}_1 &= \frac{2}{n\alpha(n)} \int_{|y|\le 1 \bigcap \partial\mathbb{R}^n_+} \frac{|y|}{|\lambda e_n - y|^n} dy \\ &\ge \frac{2}{n\alpha(n)} \int_{|y|\le 1 \bigcap \partial\mathbb{R}^n_+} \frac{|y|}{(\lambda + |y|)^n} dy\end{aligned}$$

Note that for fixed y, $\frac{|y|}{(\lambda+|y|)^n}$ is increasing when λ is decreasing to 0, so by Monotone Convergence theorem, we have

$$\begin{aligned}&\lim_{\lambda\to 0} \frac{2}{n\alpha(n)} \int_{|y|\le 1 \bigcap \partial\mathbb{R}^n_+} \frac{|y|}{(\lambda + |y|)^n} dy \\ &= \int_{|y|\le 1 \bigcap \partial\mathbb{R}^n_+} \frac{|y|}{|y|^n} dy \\ &= \int_{B_{n-1}(0,1)} \frac{|y|}{|y|^n} dy \\ &= \int_0^1 dr \int_{\partial B_{n-1}(0,r)} \frac{1}{|y|^{n-1}} dS(y) = C \int_0^1 \frac{1}{r^{n-1}} r^{n-2} dr = \infty.\end{aligned}$$

So, Du is unbounded near $x = 0$. □

Problem 10.

Suppose u is smootha nd solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.
(i) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.
(ii) Use (i) to show $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

(i) $u_{\lambda t}(x,t) = \lambda^2 u_t(\lambda x, \lambda^2 t)$ and $u_{\lambda x_i}(x,t) = \lambda u(\lambda x, \lambda^2 t)$ for each i. Then $u_{\lambda x_i x_i}(x,t) = \lambda^2 u_{x_i}(\lambda x, \lambda^2 t)$. Consequently, $\Delta u_\lambda = \lambda^2 \Delta u$ and $u_{\lambda t} - \Delta u_\lambda = \lambda^2(u_t - \Delta u)$, so u_λ solves the heat equation for all $\lambda \in \mathbb{R}$.

(ii) We differentiate $u(\lambda x, \lambda^2 t) = u(\lambda x_1, \ldots, \lambda x_n, \lambda^2 t)$ with respect to λ we get

$$\sum_k x_k u_{x_k}(\lambda x_1, \ldots, \lambda x_k, \lambda^2 t) + 2\lambda t u_t(\lambda x_1, \ldots, \lambda x_n, \lambda^2 t) = x \cdot D(\lambda x, \lambda^2 t) + 2tu_t(\lambda x, \lambda^2 t).$$

Taking $\lambda = 1$, we then have that $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$. u is smooth, so the second derivatives of $u(\lambda x, \lambda^2 t)$ are continuous, meaning the mixed partials are equal. Therefore, $v_t - \Delta v = \frac{\partial}{\partial t \partial \lambda} u(\lambda x, \lambda^2 t) - \Delta \frac{\partial}{\partial \lambda} u(\lambda x, \lambda^2 t) = \frac{\partial}{\partial \lambda \partial t} u(\lambda x, \lambda^2 t) - \frac{\partial}{\partial \lambda} \Delta u(\lambda x, \lambda^2 t) = \frac{\partial}{\partial \lambda}(u_{\lambda t} - \Delta u_\lambda) = 0$, since u_λ satisfies the heat equation for all λ. Thus v does as well.

Problem 11: Assume $n = 1$ and $u(x,t) = v(\frac{x^2}{t})$.
a) Show

$$u_t = u_{xx}$$

if and only if

(2) $$4zv''(z) + (2 + z)v'(z) = 0 \qquad (z > 0)$$

b) Show that the general solution of (1) is

$$v(z) = c\int_0^z e^{-s/4}s^{-1/2}ds + d$$

c) Differentiate $v(\frac{x^2}{t})$ with respect to x and select the constant c properly, so as to obtain the fundamental solution Φ for $n = 1$.

Solution:
a) Assume that $u_t = u_{xx}$. Then

$$u_t = -\frac{x^2}{t^2}v'\left(\frac{x^2}{t}\right)$$

and

$$u_{xx} = 2v'\left(\frac{x^2}{t}\right) + 4x^2v''\left(\frac{x^2}{t}\right)$$

So $u_t = u_{xx}$ implies that

$$-\frac{x^2}{t^2}v'\left(\frac{x^2}{t}\right) = 2v'\left(\frac{x^2}{t}\right) + 4x^2v''\left(\frac{x^2}{t}\right)$$

or

$$\frac{4x^2}{t^2}v''\left(\frac{x^2}{t}\right) + \left(\frac{2}{t} + \frac{x^2}{t^2}\right)v'\left(\frac{x^2}{t}\right) = 0$$

If we let $z = \frac{x^2}{t}$, we get

$$\frac{4z}{t}v''(z) + \left(\frac{2}{t} + \frac{z}{t}\right)v'(z) = 0$$

Multiplying this equation by t gives the desired equality.
For the other direction, reverse the steps, and hence our proof is done.

b)

$$4zv'' + (2 + z)v' = 0$$

$\Longrightarrow$

$$\frac{v''}{v'} = -\frac{1}{2}\frac{1}{z} - \frac{1}{4}$$

$\Longrightarrow$

(by integrating) $$\log(v') = -\log\sqrt{z} - \frac{z}{4} + c$$

$\Longrightarrow$

$$v' = Cz^{-1/2}e^{-z/4}$$

$\Longrightarrow$

$$v = C\int_0^z e^{-s/4}s^{-1/2}ds + d$$

as is desired.

c)

$$v(z) = c\int_0^z e^{-s/4}s^{-1/2}ds + d$$

$\Longrightarrow$

$$v\left(\frac{x^2}{t}\right) = c\int_0^{\frac{x^2}{t}} e^{-s/4}s^{-1/2}ds + d$$

$\Longrightarrow$

$$v'\left(\frac{x^2}{t}\right) = c\frac{2x}{t}e^{-\frac{x^2}{4t}}\left(\frac{x^2}{t}\right)^{-1/2}$$

or

$$v'\left(\frac{x^2}{t}\right) = \frac{2c}{\sqrt{t}}e^{-\frac{x^2}{4t}}$$

Now we want to integrate over $\mathbb{R}$ and set the integral equal to 1. Thus we get

$$1 = \frac{2c}{\sqrt{t}}\int_{\infty}^{\infty} e^{-\frac{x^2}{4t}}dx$$

Letting $y = \frac{x}{\sqrt{4t}}$, we get $dy = (4t)^{-1/2}dx$ and substituting, we get

$$1 = \frac{2c}{\sqrt{t}}\int_{\infty}^{\infty} \sqrt{4t}e^{-y^2}dy$$

or

$$1 = 4c\int_{\infty}^{\infty} e^{-y^2}dy$$

Employing the identity $\int_{\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}$ and solving for c, we get

$$c = \frac{1}{4\sqrt{\pi}}$$

Thus,

$$\begin{aligned}\Phi(x,t) :&= v'\left(\frac{x^2}{t}\right)\\ &= \frac{2c}{\sqrt{t}}e^{-\frac{x^2}{4t}}\\ &= \frac{1}{2\sqrt{\pi t}}e^{\frac{-x^2}{4t}}\end{aligned}$$

is easily shown to solve the equation

$$\Phi_t = \Phi_{xx}$$

□

Problem 12. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \text{x}(0,\infty) \\ u = g & \text{on } \mathbb{R}^n \text{x}\{t=0\}, \end{cases}$$

where c∈ $\mathbb{R}$.

Solution: Set $v(x,t) = u(x,t)e^{Ct}$. Then, $v_t = u_t e^{Ct} + Ce^{Ct}u$ and $v_{x_i x_i} = u_{x_i x_i} e^{Ct}$.
$\Rightarrow$

$$\begin{aligned} v_t - \Delta v &= u_t e^{Ct} + Ce^{Ct}u - e^{Ct}\Delta u \\ &= e^{Ct}(u_t - \Delta u + Cu) \\ &= e^{Ct} f. \end{aligned}$$

So, v is a solution of

$$\begin{cases} v_t - \Delta v = e^{Ct} f & \text{in } \mathbb{R}^n \text{x}(0,\infty) \\ \qquad\qquad v = g & \text{on } \mathbb{R}^n \text{x}\{t=0\}, \end{cases}$$

By (17) (p.51),

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)e^{Cs} f(y,s)dyds$$

where Φ is the fundamental solution of the hear equation. Since $v(x,t) = u(x,t)e^{Ct}$, we have

$$u(x,t) = e^{Ct}\Big(\int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)e^{Cs} f(y,s)dyds \Big).$$

□

Problem 13: Given $g : [0,\infty] \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)ds, x > 0$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} & = 0 \quad in \mathbb{R}_+ \times (0,\infty) \\ \qquad\quad u & = 0 \quad on \mathbb{R}_+ \times \{t=0\}, \\ \qquad\quad u & = g \quad on \{x=0\} \times [0,\infty). \end{cases}$$

Proof. We define

$$v(x,t) = \begin{cases} u(x,t) - g(t) & x > 0, \\ -u(-x,t) + g(t) & x \le 0. \end{cases}$$

So, we have

$$v_t(x,t) = \begin{cases} u_t(x,t) - g'(t) & x > 0, \\ -u_t(-x,t) + g'(t) & x \le 0, \end{cases}$$

and

$$v_{xx}(x,t) = \begin{cases} u_{xx}(x,t) & x > 0, \\ -u_{xx}(-x,t) & x \le 0. \end{cases}$$

Hence,

$$\begin{cases} v_t(x,t) - v_{xx}(x,t) = \begin{cases} -g'(t) & x > 0, \\ g'(t) & x \le 0. \end{cases} \\ v(x,0) = 0, \\ v(0,t) = 0. \end{cases}$$

By formula (13) on page 49, we get

$$v(x,t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left\{ \int_{-\infty}^0 e^{\frac{-(y-x)^2}{4(t-s)}} g'(s) dy ds - \int_0^\infty e^{\frac{-(y-x)^2}{4(t-s)}} g'(s) dy ds \right\}$$

Note that(page 46 Lemma)

$$\int_{-\infty}^\infty \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-(y-x)^2}{4(t-s)}} dy = 1,$$

so when $x > 0$, we let $y - x = -z$ and obtain

$$\begin{aligned} u(x,t) &= v(x,t) + g(t) \\ &= v(x,t) + \int_0^t g'(s) ds \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-(y-x)^2}{4(t-s)}} dy \\ &= 2\int_0^t \frac{1}{\sqrt{4\pi}} (t-s)^{-\frac{1}{2}} \int_{-\infty}^0 e^{\frac{-(y-x)^2}{4(t-s)}} dy\, g'(s) ds \\ &= \int_0^t \frac{1}{\sqrt{\pi}} (t-s)^{-\frac{1}{2}} \int_x^\infty e^{\frac{-z^2}{4(t-s)}} dz\, dg(s) \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} u(x,t) = &\frac{1}{\sqrt{\pi}} (t-s)^{-1/2} \int_x^\infty e^{\frac{-z^2}{4(t-s)}} dz\, g(s)\big|_{s=0}^{s=t} \\ &- \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{-z^2}{4(t-s)}} dz \\ &- \int_0^t g(s) \frac{1}{\sqrt{\pi}} (t-s)^{-1/2} ds \int_x^\infty e^{\frac{-z^2}{4(t-s)}} \frac{-z^2}{4(t-s)^2} dz \\ = &\mathbf{I}_1 - \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{-z^2}{4(t-s)}} dz \\ &+ \int_0^t g(s) \frac{1}{\sqrt{\pi}} (t-s)^{-1/2} ds \int_x^\infty \frac{-z}{2(t-s)} de^{\frac{-z^2}{4(t-s)}} \\ = &\mathbf{I}_1 - \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{-z^2}{4(t-s)}} dz \\ &+ \int_0^t g(s) \frac{1}{\sqrt{4\pi}} (t-s)^{-3/2} ds\, (-z)\, e^{\frac{-z^2}{4(t-s)}} \big|_{z=x}^{z=\infty} \\ &+ \int_0^t g(s) \frac{1}{\sqrt{\pi}} \frac{1}{2} (t-s)^{-3/2} ds \int_x^\infty e^{\frac{-z^2}{4(t-s)}} dz \\ = &\mathbf{I}_1 + \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s) ds. \end{aligned}$$

Now, we focus on $\mathbf{I}_1$ and define w^2 to be $\frac{z^2}{4\epsilon}$,

$$\begin{aligned}\mathbf{I}_1 &= \lim_{\epsilon\to 0^+} \frac{1}{\sqrt{\pi}}\epsilon^{-1/2}\int_x^\infty e^{\frac{-z^2}{4\epsilon}}dz\, g(t-\epsilon)\\ &= g(t)\lim_{\epsilon\to 0^+}\frac{1}{\sqrt{\pi}}\int_{x^2/4\epsilon}^\infty 2e^{-w^2}dw = 0.\end{aligned}$$

Thus, we proved

$$u(x,t) = \frac{x}{\sqrt{4\pi}}\int_0^t \frac{1}{(t-s)^{3/2}}e^{\frac{-x^2}{4(t-s)}}g(s)ds, x > 0.$$

Next, we need to show that

$$\lim_{x\to 0^+} u(x,t) = g(t).$$

Note that for any fixed $\delta > 0$.

$$\begin{aligned}\lim_{x\to 0^+} u(x,t) &= \lim_{x\to 0^+}\frac{x}{\sqrt{4\pi}}\int_{t-\delta}^t \frac{1}{(t-s)^{3/2}}e^{\frac{-x^2}{4(t-s)}}g(s)ds\\ &\quad+\lim_{x\to 0^+}\frac{x}{\sqrt{4\pi}}\int_0^{t-\delta}\frac{1}{(t-s)^{3/2}}e^{\frac{-x^2}{4(t-s)}}g(s)ds\\ &= g(t)\lim_{x\to 0^+}\frac{x}{\sqrt{4\pi}}\int_{t-\delta}^t\frac{1}{(t-s)^{3/2}}e^{\frac{-x^2}{4(t-s)}}ds\\ &= g(t)\lim_{x\to 0^+}\frac{x}{\sqrt{4\pi}}\int_0^\delta \frac{1}{s^{3/2}}e^{\frac{-x^2}{4s}}ds\end{aligned}$$

For fixed x, we let $s = x^2/w^2$ and get

$$\begin{aligned}\lim_{x\to 0^+} u(x,t) &= g(t)\lim_{x\to 0^+}\frac{x}{2\sqrt{\pi}}\int_\infty^{x^2/\delta}\frac{w^3}{x^3}e^{\frac{-w^2}{4}}\frac{-2x^2}{w^3}dw\\ &= g(t)\lim_{x\to 0^+}\frac{1}{\sqrt{\pi}}\int_{x^2/\delta}^\infty e^{\frac{-w^2}{4}}dw\\ &= g(t)\frac{1}{\sqrt{\pi}}\int_0^\infty e^{\frac{-w^2}{4}}dw = g(t).\end{aligned}$$

Hence, we are done. □

Problem 14. We say $v \in C_1^2(U_T)$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \le 0 \quad \text{in } U_T.$$

(a) Prove for a subsolution v that

$$v(x,t) \le \frac{1}{4r^n}\int\int_{E(x,t;r)} v(y,s)\frac{|x-y|^2}{(t-s)^2}\, dyds$$

for all $E(x,t;r) \subset U_T$.

(b) Prove that therefore $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$

Solution.

(a) We may well assume upon translating the space and time coordinates that $x = 0$ and $t = 0$. As in the proof of Theorem 3, set

$$\phi(r) := \frac{1}{r^n}\int\int_{E(r)} v(y,s)\frac{|y|^2}{s^2}dyds,$$
$$\psi(y,s) := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r$$

and derive

$$\phi'(r) \geq \frac{1}{r^{n+1}}\int\int_{E(r)} -4n\Delta v\psi - \frac{2n}{s}\sum_{i=1}^{n} v_{y_i}y_i\, dyds$$
$$= \sum_{i=1}^{n}\frac{1}{r^{n+1}}\int\int_{E(r)} 4nv_{y_i}\psi_{y_i} - \frac{2n}{s}v_{y_i}y_i\, dyds = 0.$$

For $0 < \epsilon < r$,

$$\int_{\epsilon}^{r} \phi'(z)dz = \phi(r) - \phi(\epsilon) \geq 0.$$

Hence, $\phi(r) \geq \lim_{\epsilon\to 0}\phi(\epsilon) = v(0,0)\cdot\lim_{\epsilon\to 0}\frac{1}{\epsilon^n}\int\int_{E(\epsilon)}\frac{|y|^2}{s^2}dyds = 4v(0,0)$, and the statement follows.

(b) Suppose there exists a point $(x_0,t_0) \in U_T$ with $u(x_0,t_0) = M := \max_{\bar{U}_T} u$. Then for all sufficiently small $r > 0$, $E(x_0,t_0;r) \subset U_T$. Using the result proved above, we deduce

$$M = v(x_0,t_0) \leq \frac{1}{4r^n}\int\int_{E(x_0,t_0;r)} v(y,s)\frac{|x-y|^2}{(t-s)^2}\, dyds \leq M,$$

since

$$1 = \frac{1}{4r^n}\int\int_{E(x_0,t_0;r)}\frac{|x_0-y|^2}{(t_0-s)^2}\, dyds.$$

Conclude that $u|_{E(x_0,t_0;r)} = M$. The argument used in the proof of Theorem 4 will finish the proof.

□

Problem 15.
(a) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F,G.
(b) Using the change of variables $\xi = x + t, \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.
(c) Use (a),(b) to rederive d'Alembert's formula.

Solution:
(a)
$u_{xy} = 0 \Rightarrow u_x = f(x) \Rightarrow u(x,y) = \int f(x)dx + G(y)$
$u_{yx} = 0 \Rightarrow u_y = g(y) \Rightarrow u(x,y) = \int g(y)dy + F(x)$

This implies $u(x,y) = F(x) + G(y)$.

(b)
$x = \frac{\xi+\eta}{2}, y = \frac{\xi-\eta}{2}$
Define $\tilde{u} := u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2})$

$$\tilde{u}_\xi = \frac{1}{2}u_x + \frac{1}{2}u_t \qquad \text{and} \qquad \tilde{u}_{\xi\eta} = \frac{1}{4}u_{xx} - \frac{1}{4}u_{xt} + \frac{1}{4}u_{tx} - \frac{1}{4}u_{tt} = \frac{1}{4}(u_{xx} - u_{tt})$$

Hence, $\tilde{u}_{\xi\eta} = 0 \Leftrightarrow u_{tt} - u_{xx} = 0$.

(c)
By (b), $u_{tt} - u_{xx} = 0 \Rightarrow u_{\xi\eta} = 0$, and $u(\xi,\eta) = F(\xi) + G(\eta)$ by (a)
,i.e, $u(x,y) = F(x+t) + G(x-t)$.
Since $u(x,0) = g, u_t(x,0) = h$,

$$\text{(3)} \qquad u(x,0) = F(x) + G(x) = g(x),$$
$$u_t(x,0) = F'(x) - G'(x) = h(x)$$

Integration $\Rightarrow$

$$\text{(4)} \qquad F(x) - G(x) = \int_0^x h(y)dy + C, \quad \text{C:constant.}$$

$$(2) + (3); \qquad F(x) = \frac{1}{2}(g(x) + \int_0^x h(y)dy + C)$$
$$(2) - (3); \qquad G(x) = \frac{1}{2}(g(x) - \int_0^x h(y)dy - C)$$

Thus,

$$\begin{aligned} u(x,y) = F(x+t) + G(x-t) &= \frac{1}{2}(g(x+t) + \int_0^{x+t} h(y)dy + C) + \frac{1}{2}(g(x-t) - \int_0^{x-t} h(y)dy - C) \\ &= \frac{1}{2}(g(x+t) + \int_0^{x+t} h(y)dy + C + g(x-t) + \int_{x-t}^0 h(y)dy - C) \\ &= \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \qquad (x \in \mathbb{R}, t \geq 0). \end{aligned}$$

□

Problem 16.

Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations:

$$\mathbf{E}_t = \operatorname{curl}\mathbf{B}$$
$$\mathbf{B}_t = -\operatorname{curl}\mathbf{E}$$
$$\operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0$$

Show that $u_{tt} - \Delta u = 0$ where $u = B^i$ or E^i for $i = 1,2,3$.

Solution.

$$\begin{aligned}\operatorname{curl}(\operatorname{curl}\mathbf{E}) &= \operatorname{curl}(-\mathbf{B}_t)\\ &= \left(-\frac{\partial^2 B^3}{\partial y\partial t}+\frac{\partial^2 B^2}{\partial z\partial t},-\frac{\partial^2 B^3}{\partial x\partial t}+\frac{\partial^2 B^1}{\partial z\partial t},-\frac{\partial^2 B^2}{\partial x\partial t}+\frac{\partial B^1}{\partial y\partial t}\right)\\ &= -\frac{\partial}{\partial t}\operatorname{curl}\mathbf{B}\\ &= -\frac{\partial}{\partial t}\mathbf{E}_t\\ &= -\frac{\partial^2\mathbf{E}}{\partial t^2}\end{aligned}$$

However, we also know that $\operatorname{curl}(\operatorname{curl}\mathbf{E}) = \nabla(\operatorname{div}\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$. Then E^i satisfies $u_{tt} - \Delta u = 0$ for $i = 1, 2, 3$.

Similarly, $\operatorname{curl}(\operatorname{curl}\mathbf{B}) = \operatorname{curl}\mathbf{E}_t = -\frac{\partial^2\mathbf{B}}{\partial t^2}$, and $\operatorname{curl}(\operatorname{curl}\mathbf{B}) = \nabla(\operatorname{div}\mathbf{B}) - \nabla^2\mathbf{B} = -\nabla^2\mathbf{B}$, so B^i satisfies $u_{tt} - \Delta u = 0$ for $i = 1, 2, 3$.

Problem 17.(Equipartition of energy) Let $u \in C^2(\mathbb{R}\times[0,\infty))$ solve the initial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in} \quad \mathbb{R}\times(0,\infty)\\ u = g; \quad u_t = h & \text{on} \quad \mathbb{R}\times\{t=0\}.\end{cases}$$

Suppose g, h have compact support. The kinetic energy is $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ and the potential energy is $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$. Prove

(i) $\quad k(t) + p(t) \quad$ is constant in t.

(ii) $\quad k(t) = p(t) \quad$ for all large enough times t.

Proof. (i.) We define $e(t) = k(t) + p(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left(u_t^2 + u_x^2\right)dx$. Since g, h have compact support, so we have

$$\begin{aligned}\frac{d\,e(t)}{dt} &= \frac{1}{2}\int_{-\infty}^{\infty} 2u_t u_{tt} + 2u_x u_{xt}\,dx\\ &\int_{-\infty}^{\infty} u_t u_{tt}\,dx - \int_{-\infty}^{\infty} u_{xx}u_t\,dx\\ &= \int_{-\infty}^{\infty} u_t\left(u_{tt} - u_{xx}\right)\,dx = 0.\end{aligned}$$

Hence, $e(t) \equiv e(0)$.
(ii.)By d'Alembert's formula on page 68, we have

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

So,

$$u_t = \frac{1}{2}\left[g'(x+t) - g'(x-t)\right] + \frac{1}{2}\left[h(x+t) + h(x-t)\right],$$

and

$$u_x = \frac{1}{2}\left[g'(x+t) + g'(x-t)\right] + \frac{1}{2}\left[h(x+t) - h(x-t)\right].$$

We assume that there exists a positive constant M so that $[-M, M] \supseteq supp(g')$ and $[-M, M] \supseteq supp(h)$.
Note that for a fixed $t > M$, $-M \leq x - t \leq M \Leftrightarrow 0 < t - M \leq x \leq t + M$ and $-M \leq x + t \leq M \Leftrightarrow -t - M \leq x \leq -t + M < 0$.
Thus, when $t > M$:

(a) $0 < t - M \leq x \leq t + M$.
Then we have

$$h(x + t) = g(x + t) = 0.$$

So,

$$u_t^2 = \frac{1}{4}g'(x - t)^2 + \frac{1}{4}h(x - t)^2 - \frac{1}{2}g'(x - t)h(x - t) = u_x^2.$$

(b) $-t - M \leq x \leq -t + M < 0$.
Then,

$$u_t^2 = \frac{1}{4}g'(x + t)^2 + \frac{1}{4}h(x + t)^2 + \frac{1}{2}g'(x + t)h(x + t) = u_x^2.$$

(c) Otherwise

$$g'(x + t) = g'(x - t) = h(x + t) = h(x - t) = 0.$$

So, combining all the cases, it is obvious that when $t > M$, $k(t) = p(t)$. □

Problem 18. Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where g, h are smooth and have compact support. Show there exists a constant C such that

$$|u(x, t)| \leq C/t \quad (x \in \mathbb{R}^3, t > 0).$$

Solution.

From the conditions it follows that there exist $R, M > 0$ such that spt g, spt $h \subset B(0, R)$ and $g(y) \leq M$, $|Dg(y)| \leq M$, $h(y) \leq M$ for any $y \in \mathbb{R}^3$. Kirchhoff's formula gives the solution of the initial-value problem:

$$u(x, t) = \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y) \cdot (y - x) \, dS(y).$$

Denote by Σ the intersection $\partial B(x, t) \cap B(0, R)$. Observe that the area of Σ is not greater than the area of the sphere $\partial B(0, R)$. Then, for $t > 0$, we obtain

$$\begin{aligned} \left| \fint_{\partial B(x,t)} th(y) + Dg(y) \cdot (y - x) \, dS(y) \right| &= \frac{1}{4\pi t^2} \left| \int_{\partial B(x,t) \cap B(0,R)} th(y) + Dg(y) \cdot (y - x) \, dS(y) \right| \\ &\leq \frac{1}{4\pi t^2} \int_{\partial B(x,t) \cap B(0,R)} t \cdot |h(y)| + |Dg(y)| \cdot |y - x| \, dS(y) \\ &\leq \frac{1}{4\pi t^2} \cdot 4\pi R^2 \cdot (tM + tM) = \frac{2R^2 M}{t}. \end{aligned}$$

For $t > 1$, using the same argument, we get

$$\left| \fint_{\partial B(x,t)} g(y)\, dS(y) \right| = \frac{1}{4\pi t^2} \left| \int_{\partial B(x,t) \cap B(0,R)} g(y)\, dS(y) \right| \le \frac{1}{4\pi t^2} \cdot 4\pi R^2 \cdot M = \frac{R^2 M}{t^2} \le \frac{R^2 M}{t}.$$

Notice now that the area Σ is not greater than the area of the sphere $\partial B(x,t)$. Then for $0 < t \le 1$,

$$\left| \fint_{\partial B(x,t)} g(y)\, dS(y) \right| = \frac{1}{4\pi t^2} \left| \int_{\partial B(x,t) \cap B(0,R)} g(y)\, dS(y) \right| \le \frac{1}{4\pi t^2} \cdot 4\pi t^2 \cdot M \le \frac{M}{t}.$$

Without loss of generality, we can take $R > 1$. Then, combining the estimates obtained above, we conclude $|u(x,t)| \le \frac{3R^2 M}{t}$. □

Evans PDE Solutions, Chapter 5

Alex: 4, Helen: 5, Rob H.: 1

Problem 1.

Suppose $k \in \{0, 1, \ldots\}$, $0 < \gamma < 1$. Prove $C^{k,\gamma}(\bar{U})$ is a Banach space.

Solution:

1. First we show that $\|\cdot\|_{C^{k,\gamma}(\bar{U})}$ is a norm, where we recall that

$$\|u\|_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})},$$

and

$$[u]_{C^{0,\gamma}(\bar{U})} = \sup_{x \ne y \in U} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\}.$$

For the sake of opaqueness we now omit subscripts on all norms unless it is unclear from context.

2. For any $\lambda \in \mathbb{R}$ we have first

$$[\lambda u] = \sup_{x,y \in U} \frac{|\lambda u(x) - \lambda u(y)|}{|x - y|^\gamma} = |\lambda| \sup_{x,y \in U} \frac{|u(x) - u(y)|}{|x - y|^\gamma} = |\lambda|\, [u],$$

and certainly

$$\|D^\alpha(\lambda u)\|_{C(\bar{U})} = \|\lambda D^\alpha u\| = |\lambda| \cdot \|D^\alpha u\|.$$

So

$$\begin{aligned} \|\lambda u\| &= \sum_{|\alpha| \le k} \|D^\alpha(\lambda u)\| + \sum_{|\alpha| = k} [D^\alpha(\lambda u)] \\ &= |\lambda| \sum_{|\alpha| \le k} \|D^\alpha u\| + |\lambda| \sum_{|\alpha| = k} [D^\alpha u] \\ &= |\lambda| \cdot \|u\|. \end{aligned}$$

3. If $u = 0$ it is obvious that $\|u\| = 0$. On the other hand, $\|u\| = 0$ implies that

$$\|D^\alpha u\|_{C(\bar{U})} = 0$$

for every $|\alpha| \leq k$. In particular this is true for $\alpha = 0$ so that the supremum of $D^0 u = u$ on U is 0, i.e. $u \equiv 0$.

4. Finally we must prove the triangle inequality. We know the triangle inequality is true for the sup norm $\|\cdot\|_{C(\bar{U})}$. We can also see that for any α which makes sense

$$[D^\alpha(u+v)] = [D^\alpha u + D^\alpha v] \leq [D^\alpha u] + [D^\alpha v]\,.$$

Therefore we can easily conclude

$$\begin{aligned} \|u+v\| &= \sum_{|\alpha|\leq k} \|D^\alpha(u+v)\| + \sum_{|\alpha|=k} [D^\alpha(u+v)] \\ &\leq \sum_{|\alpha|\leq k} (\|D^\alpha u\| + \|D^\alpha v\|) + \sum_{|\alpha|=k} ([D^\alpha u] + [D^\alpha v]) \\ &= \|u\| + \|v\|. \end{aligned}$$

5. We need only show that $C^{k,\gamma}(U)$ is complete. So let $\{u_m\}$ be a Cauchy sequence. Then $\{u_m(x)\{$ is a Cauchy sequence for every x, so define u to be the pointwise limit of the u_m. Now if V is any bounded subset of U, then $\bar{V}$ is compact, so that $u_m \Rightarrow u$ uniformly on any V. Since the u_m are uniformly continuous on $\bar{V}$ by assumption, this implies that u is uniformly continuous on $\bar{V}$ as well (and so, *a fortiori* $u \in C(U)$). Therefore $u \in C(\bar{U})$.

What we would really like would be to have $u \in C^k(\bar{U})$. But similar arguments show that u has derivatives $D^\alpha u$ for all $|\alpha| \leq k$ on U by restricting first to bounded subsets of U to find the derivatives and then using uniform convergence on these subsets to show the derivatives must also be uniformly continuous on bounded subsets since the $D^\alpha u_m$ were.

This leaves us with only showing that the norm of u is finite, so that in fact $u \in C^{k,\gamma}(U)$. But for every n we have

$$\begin{aligned} \|u_n - u\| &= \sum_{|\alpha|\leq k} \sup_{x\in U} |D^\alpha u_n(x) - D^\alpha u(x)| + \sum_{|\alpha|=k} \sup_{x,y\in U} \frac{|D^\alpha u_n(x) - D^\alpha u_n(y) - D^\alpha u(x) + D^\alpha u(y)|}{|x-y|^\gamma} \\ &= \lim_{m\Rightarrow\infty} \left(\sum_{|\alpha|\leq k} \sup_{x\in U} |D^\alpha u_n(x) - D^\alpha u_m(x)| + \sum_{|\alpha|=k} \sup_{x,y\in U} \frac{|D^\alpha u_n(x) - D^\alpha u_n(y) - D^\alpha u_m(x) + D^\alpha u_m(y)|}{|x-y|^\gamma} \right) \\ &= \lim_{m\Rightarrow\infty} \|u_n - u_m\|. \end{aligned}$$

In particular, since $\{u_m\}$ is Cauchy there is some N so that $n, m \geq N$ implies $\|u_n - u_m\| \leq 1$. Letting m approach ∞, this implies that $\|u_N - u\| < 1$. Now the triangle inequality applies to give

$$\|u\| \leq \|u_N - u\| + \|u_N\| < 1 + \|u_N\| < \infty.$$

□

Problem 4.

Assume U is bounded and $U \subset\subset \bigcup_{i=1}^N V_i$. Show there exist C^∞ functions ζ_i $(i = 1, \ldots, N)$ such that

$$\begin{cases} 0 \leq \zeta_1 \leq 1, \operatorname{supp} \zeta_i \subset V_i & i = 1, \ldots, N \\ \sum_{i=1}^N \zeta_i = 1 & \text{on } U. \end{cases}$$

The functions $\{\zeta_i\}_1^N$ for a *partition of unity*.

Solution. Assume U is bounded and $U \subset\subset \bigcup_{i=1}^{N} V_i$. Without loss of generality, we may assume that the V_i are open, for if they are not, we can replace V_i by its interior. We note that, since U is bounded, $\overline{U}$ is compact. Each $x \in U$ has a compact neighbourhood N_x contained in V_i for some i. Then $\{N_x^\circ\}$ is an open cover of $\overline{U}$, which then has a finite subcover $N_{x_1}^\circ, \ldots, N_{x_n}^\circ$. We now let F_i be the union of the N_{x_k} contained in V_i. F_i is the compact since it is the finite union of compact sets. The C^∞ version of Urysohn's Lemma (Folland, p.245) allows us to find smooth functions $\xi_1, \ldots, \xi_N$ such that $\xi_i = 1$ on F_i and $\mathrm{supp}(\xi_i) \subset V_i$. Since the F_i cover U, $U \subset \{x : \sum_1^n \xi_i(x) > 0\}$ and we can use Urysohn again to find $\zeta \in C^\infty$ with $\zeta = 1$ on $\overline{U}$ and $\mathrm{supp}(\zeta) \subset \{x : \sum_1^n \xi_i(x) > 0\}$. Now, we let $\xi_{N_1} = 1 - \zeta$, so $\sum_1^{N+1} \xi_i > 0$ everywhere. We then take

$$\zeta_i = \frac{\xi_i}{\sum_1^{N+1} \xi_j}$$

as our partition of unity.

Problem 5 (Helen) Prove that if $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$, then u is equal a.e. to an absolutely continuous function, and u' which exists a.e. belongs to $L^p(0,1)$.

Proof. Since $u \in W^{1,p}(0,1)$, so by definition on page 242 and 244, we have some function $v \in L^p(0,1)$ such that

$$\int_{(0,1)} u\, D\phi dx = -\int_{(0,1)} v\phi dx, \quad \forall \phi \in C_c^\infty((0,1)).$$

Note that $v \in L^p(0,1)$, so by Hölder's inequality, we have $\|v\|_{L^1} \leq \|v\|_{L^p} \|1\|_{L^q} < \infty$, which means $v \in L^1(0,1)$. Thus, we can define function $f(x)$ on $(0,1)$ by the following formula

$$f(x) = u(\frac{1}{2}) + \int_{\frac{1}{2}}^{x} v(t)dt, \quad \forall x \in (0,1).$$

According to the Fundamental Theorem of Calcalus, f is absolutely continuous. Now we will prove $u = f$ a.e.

By the definition of f, we have $f' = v$ a.e. So for any $\phi \in C_c^\infty((0,1))$ we get

$$\int_{(0,1)} f\, D\phi dx = -\int_{(0,1)} f'\phi dx = -\int_{(0,1)} v\phi dx.$$

Therefore,

$$\int_{(0,1)} (f - u)\, D\phi dx = 0 \quad \forall \phi \in C_c^\infty((0,1)),$$

which means $u = f + \mathtt{const}$. And note that $u(\frac{1}{2}) = f(\frac{1}{2})$, hence $u = f$ a.e. So u' exists a.e. and satisfy $u' = v$ a.e., so $u' \in L^p(0,1)$. □