Folland: Real Analysis, Chapter 1
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Problem 1.5
If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: Show that the latter object is a σ-algebra.)

Solution: Let $\mathcal{N}$ denote the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

$$\mathcal{N} = \bigcup_{\mathcal{F}\subset\mathcal{E},\ \mathcal{F} \text{ countable}} M(\mathcal{F}).$$

When $\mathcal{F} \subset \mathcal{E}$, the σ-algebra generated by $\mathcal{F}$ is contained in the σ-algebra generated by $\mathcal{E}$ ie $M(\mathcal{F}) \subset M(\mathcal{E}) = \mathcal{M}$, so we have $\mathcal{N} \subset \mathcal{M}$.

For the reverse containement, we first observe that $\mathcal{E} \subset \mathcal{N}$. Indeed, if $\mathcal{E} = \{A_\alpha\}_{\alpha\in J}$ where J is some index set, then

$$\mathcal{E} \subset \bigcup_{\alpha\in J} M(\{A_\alpha\}) \subset \mathcal{N}.$$

Next, we will show that $\mathcal{N}$ is a σ-algebra. Since $M(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$, we will have $\mathcal{M} = M(\mathcal{E}) \subset \mathcal{N}$ and this will complete the proof.

Let $A \in \mathcal{N}$. Then $A \in M(\mathcal{F})$ for some countable $\mathcal{F} \subset \mathcal{E}$. Therefore $A^c \in M(\mathcal{F}) \subset \mathcal{N}$. Next, consider $\cup_{i=1}^\infty A_i$ where $A_i \in \mathcal{N}$. Then each $A_i \in M(\mathcal{F}_i)$ for some countable $\mathcal{F}_i \subset \mathcal{E}$. Since $\cup_{j=1}^\infty \mathcal{F}_j$ is countable, we have $A_i \in M(\cup_{j=1}^\infty \mathcal{F}_j) \subset \mathcal{N}$ for each A_i. Hence $\cup_{i=1}^\infty A_i \in M(\cup_{j=1}^\infty \mathcal{F}_j) \subset \mathcal{N}$.

Problem 1.12
Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

a. *If $E, F \in \mathcal{M}$ and $\mu(E\Delta F) = 0$, then $\mu(E) = \mu(F)$.*

b. *Say that $E \sim F$ if $\mu(E\Delta F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.*

c. *For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.*

Solution:
(a) First, notice

$$0 = \mu(E\Delta F) = \mu(E\backslash F \ \cup \ F\backslash E) = \mu(E\backslash F) + \mu(F\backslash E).$$

Hence $\mu(E\backslash F) = \mu(F\backslash E) = 0$. It follows that

$$\mu(E) = \mu(E\backslash F) + \mu(E \cap F) = \mu(E \cap F),$$

$$\mu(F) = \mu(F \backslash E) + \mu(F \cap E) = \mu(E \cap F).$$

Therefore, $\mu(E) = \mu(F)$.

(b) It is clear that $\sim$ is reflexive: $\mu(E \Delta E) = \mu(\emptyset) = 0$. It is also easy to see that $\sim$ is symmetric: if $E \sim F$, then $\mu(E \Delta F) = \mu(F \Delta E) = 0$, so $F \sim E$.

It only remains to show that $\sim$ is transitive. Suppose $E \sim F$ and $F \sim G$. Then $\mu(E \cap F^c) + \mu(F^c \cap E) = 0$ and $\mu(F \cap G^c) + \mu(F^c \cap G) = 0$. Therefore,

$$\begin{aligned}
\mu(E \Delta G) &= \mu(E \cap G^c) + \mu(E^c \cap G) \\
&= \mu(E \cap G^c \cap F^c) + \mu(E^c \cap G \cap F) + \mu(E \cap G^c \cap F) + \mu(E^c \cap G \cap F^c) \\
&\leq \mu(E \cap F^c) + \mu(E^c \cap F) + \mu(F \cap G^c) + \mu(F^c \cap G) \\
&= 0 + 0 = 0.
\end{aligned}$$

Since $\mu(E \Delta G) \geq 0$, we must have $\mu(E \Delta G) = 0$.

(c) We show the triangle inequality. Given $E, F, G \in \mathcal{M}$, then

$$\begin{aligned}
\rho(E, G) &= \mu(E \Delta G) \\
&= \mu(E \cap G^c \cap F^c) + \mu(E \cap G^c \cap F) + \mu(G \cap E^c \cap F) + \mu(G \cap E^c \cap F^c) \\
&= (\mu(E \cap G^c \cap F^c) + \mu(E \cap G \cap F^c) + \mu(F \cap G^c \cap E^c) + \mu(F \cap G \cap E^c)) \\
&+ (\mu(F \cap G^c \cap E^c) + \mu(F \cap G^c \cap E) + \mu(G \cap E^c \cap F^c) + \mu(G \cap E \cap F^c)). \\
&= \rho(E \Delta F) + \mu(F \Delta G) = \rho(E, F) + \rho(F, G).
\end{aligned}$$

Problem 1.16

Let (X, M, μ) be a measure space. A set $E \subset X$ is called locally measurable if $E \cap A \in M$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\tilde{M}$ be the collection of all locally measurable sets. Clearly $M \subset \tilde{M}$; if $M = \tilde{M}$, then μ is called saturated.

a. *If μ is σ-finite, then μ is saturated.*

b. *$\tilde{M}$ is a σ-algebra.*

c. *Define $\tilde{\mu}$ on $\tilde{M}$ by $\tilde{\mu}(E) = \mu(E)$ if $E \in M$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\tilde{M}$, called the saturation of μ.*

d. *If μ is complete, so is $\tilde{\mu}$*

e. *Suppose that μ is semifinite. For $E \in \tilde{M}$, define $\underline{\mu}(E) = \sup\{\mu(A) : A \in M \text{ and } A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\tilde{M}$ that extends μ.*

f. *Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and M the σ-algebra of countable or co-countable sets in X. Let μ_0 be the counting measure on $\mathcal{P}(X_1)$, and define μ on M by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on M, $\tilde{M} = \mathcal{P}(X)$, and in the notation of parts (c) and (e), $\tilde{\mu} \neq \underline{\mu}$.*

Solution:

(a) Suppose μ is σ-finite. Let $A \in \tilde{M}$, and let $X = \cup_{j=1}^{\infty} E_j$ where $E_j \in M$ and $\mu(E_j) < \infty$. Notice

$$A = A \cap \left(\bigcup_{j=1}^{\infty} E_j \right) = \bigcup_{j=1}^{\infty} A \cap E_j.$$

Each $E_j \cap A \in M$ since A is locally measurable, and since A is a countable union of these sets, we have $A \in M$. Hence $\tilde{M} \subset M$ and μ is saturated.

(b) Let $E \in \tilde{M}$. Take any $A \in M$ such that $\mu(A) < \infty$. Then

$$E^c \cap A = A \cap (E \cap A)^c = (A^c \cup (E \cap A))^c \in M.$$

Hence $E^c \in \tilde{M}$. Next, consider $\cup_{i=1}^{\infty} E_i$, where each $E_i \in \tilde{M}$. Then for all $A \in M$ such that $\mu(A) < \infty$, we have

$$\left(\bigcup_{i=1}^{\infty} E_i \right) \cap A = \bigcup_{i=1}^{\infty} (E_i \cap A) \in M.$$

Hence $\tilde{M}$ is a σ-algebra.

(c) We show $\tilde{\mu}$ is a measure on $\tilde{M}$. We have $\tilde{\mu} : \tilde{M} \to [0, \infty]$ and $\tilde{\mu}(\emptyset) = \mu(\emptyset) = 0$, so it only remains to show countable additivity. Let $\{E_j\}_{j=1}^{\infty}$ denote a sequence of disjoint sets in $\tilde{M}$. We partition

$$\bigcup_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} A_j \right) \cup \left(\bigcup_{j=1}^{\infty} F_j \right),$$

where $A_j, F_j \in \{E_j\}_{j=1}^{\infty} \cup \{\emptyset\}$, $A_j \in M$, and $F_j \notin M$ or $F_j = \emptyset$. By not selecting the same E_k twice, we can make sure that the elements of $\{A_1, A_2, \ldots, F_1, F_2, \ldots\}$ are all pairwise disjoint.

Case 1: $\cup_{j=1}^{\infty} F_j = \emptyset$. Then $\tilde{\mu}(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \tilde{\mu}(E_j)$.

Case 2: $\cup_{j=1}^{\infty} F_j \notin M$. Then some $E_k \notin M$, and $\tilde{\mu}(E_k) = \infty$. Furthermore, $(\bigcup A_j) \cup (\bigcup F_j) \notin M$: indeed, if $(\bigcup A_j) \cup (\bigcup F_j) \in M$, then

$$\bigcup_{j=1}^{\infty} F_j = (((\bigcup_{j=1}^{\infty} A_j) \cup (\bigcup_{j=1}^{\infty} F_j))^c \cup (\bigcup_{j=1}^{\infty} A_j))^c \in M.$$

Therefore, we have

$$\tilde{\mu}(\bigcup_{j=1}^{\infty} E_j) = \tilde{\mu} \left(\left(\bigcup_{j=1}^{\infty} A_j \right) \cup \left(\bigcup_{j=1}^{\infty} F_j \right) \right) = \infty = \sum_{j=1}^{\infty} \tilde{\mu}(E_j).$$

Case 3: $\bigcup F_j \neq \emptyset$ and $\bigcup F_j \in M$. In this case we have

$$\tilde{\mu}(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(A_j) + \mu(\bigcup_{j=1}^{\infty} F_j).$$

Hence we need to show that $\mu(\bigcup F_j) = \infty$. Suppose $\mu(\bigcup F_j) < \infty$. Choose F_j such that $F_j \neq \emptyset$. Then since F_j is locally measurable, we have

$$F_j = F_j \cap \left(\bigcup_{j=1}^{\infty} F_j \right) \in M.$$

This contradiction shows that $\tilde{\mu}(\bigcup_{j=1}^{\infty} E_j) \geq \mu(\bigcup F_j) = \infty$. Since there is some $E_k \notin M$, we have $\sum \tilde{\mu}(E_j) = \infty$.

We now prove that $\tilde{\mu}$ is saturated. Let $E \subset X$ be such that $E \cap \tilde{A} \in \tilde{M}$ when $\tilde{\mu}(\tilde{A}) < \infty$. Choose $A \in M$ such that $\mu(A) < \infty$. Then $\tilde{\mu}(A) < \infty$, and hence $E \cap A \in \tilde{M}$. But then $(E \cap A) \cap A \in M$. Therefore, $E \cap A \in M$ and $E \in \tilde{M}$. It follows that $\tilde{\mu}$ is saturated.

(d) Suppose μ is complete. If $E \in \tilde{M}$ is such that $\mu(E) = 0$, then $\tilde{\mu} < \infty$ and hence $E \in M$. Therefore, for all $A \subset E$, we have $A \in M$ by completeness of μ. Therefore, $A \in \tilde{M}$ and $\tilde{\mu}$ is complete.

(e) We prove that $\underline{\mu}$ is a measure on $\tilde{M}$. Since $\underline{\mu}(\emptyset) = \mu(\emptyset) = 0$, it remains to show countable additivity. Let $\{E_j\}$ be a collection of disjoint sets, with $E_j \in \tilde{M}$. Let $A \in M$ be such that $A \subset \bigcup E_j$.

Case 1: $\mu(A) < \infty$. Then

$$\mu(A) = \mu \left(\bigcup_{j=1}^{\infty} (A \cap E_j) \right) = \sum_{j=1}^{\infty} \mu(A \cap E_j) \leq \sum_{j=1}^{\infty} \underline{\mu}(E_i).$$

Case 2: suppose $\mu(A) = \infty$. By semifiniteness, for all $c > 0$ there exists $\tilde{A} \subset A$ such that $\tilde{A} \in M$ and $\mu(\tilde{A}) = c$. (Indeed, this is the definition of semifinite in Royden. I do not remember if it is pointed out directly in Folland, but in any case it is not hard to prove from Folland's definition.) By case 1, $c \leq \sum \underline{\mu}(E_i)$, so $\sum \underline{\mu}(E_i) = \infty$.

Therefore, $\mu(A) \leq \sum_{j=1}^{\infty} \underline{\mu}(E_i)$. Taking the supremum over all such A, we have

$$\underline{\mu} \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \underline{\mu}(E_j).$$

We now show the reverse inequality. By the definition of the supremum, there exists a sequence

$\{B_i\}_{i=1}^{\infty}$ such that $B_i \in M$, $B_i \subset E_i$, and $\underline{\mu}(E_i) \leq \mu(B_i) + \epsilon/2^i$. Therefore,

$$\begin{aligned} \sum_{i=1}^{\infty} \underline{\mu}(E_i) &\leq \sum_{i=1}^{\infty} \mu(B_i) + \epsilon \\ &= \mu\left(\bigcup_{i=1}^{\infty} B_i\right) + \epsilon \\ &\leq \underline{\mu}\left(\bigcup_{i=1}^{\infty} E_i\right) + \epsilon. \end{aligned}$$

Since this holds for all $\epsilon > 0$, we have

$$\sum_{j=1}^{\infty} \underline{\mu}(E_j) \leq \underline{\mu}\left(\bigcup_{j=1}^{\infty} E_j\right).$$

This completes the proof that $\underline{\mu}$ is a measure on $\tilde{M}$. We now prove that $\underline{\mu}$ extends μ. Let $E \in M$. For all $A \in M$ such that $A \subset E$, we have $\mu(A) \leq \mu(E)$. Since $\underline{\mu}(E)$ is the supremum over all such A, we must have $\underline{\mu}(E) \leq \mu(E)$. On the other hand, since $E \subset E$, we have

$$\mu(E) \leq \sup\{\mu(A) : A \in M \text{ and } A \subset E\}.$$

Hence $\mu(E) \leq \underline{\mu}(E)$ and $\underline{\mu}$ extends μ.

Finally, we prove that $\underline{\mu}$ is saturated. Let $E \subset X$ be such that $E \cap \tilde{A} \in \tilde{M}$ when $\underline{\mu}(\tilde{A}) < \infty$. Take any $A \in M$ such that $\mu(A) < \infty$. Then $\underline{\mu}(A) < \infty$. Hence $E \cap A \in \tilde{M}$. Therefore, $(E \cap A) \cap A \in M$. It follows that $E \cap A \in M$ and so $E \in \tilde{M}$.

(f) First, we show that μ is a measure on M. It is clear that $\mu(\emptyset) = 0$. If $\{E_i\}$ are disjoint sets in M, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu_0\left(\bigcup_{i=1}^{\infty}(E_i \cap X_1)\right) = \sum_{i=1}^{\infty} \mu_0(E_i \cap X_1) = \sum_{i=1}^{\infty} \mu(E_i).$$

Next, we show that $\tilde{M} = P(X)$. Take $E \in P(X)$. Take $A \in M$ such that $\mu(A) < \infty$. We claim that A must be countable. Indeed, suppose A is not countable. Then A^c is countable, so $X_1 \cap A^c$ is countable. Since $\mu(A) < \infty$, we must have that $X_1 \cap A$ is countable. But

$$X_1 = (X_1 \cap A) \cup (X_1 \cap A^c)$$

is uncountable. This contradiction shows that A is countable, hence $E \cap A$ is countable and $E \cap A \in M$. Therefore, $E \in \tilde{M}$.

We now show that $\tilde{\mu} \neq \underline{\mu}$. Take two elements $y_1, y_2 \in X_1$. Let $E = \{y_1, y_2\} \cup X_2$. Then $E \notin M$, so $\tilde{\mu}(E) = \infty$. However, $\underline{\mu}(E) = 2$.

Problem 1.18
Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

a. *For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.*

b. *If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \backslash E) = 0$.*

c. *If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.*

Solution:
(a) Let $E \subset X$ and $\epsilon > 0$. The definition of the induced measure μ^* is

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(A_i) : E \subset \bigcup_{i=1}^{\infty} A_i \ , \ A_i \in \mathcal{A} \right\}.$$

By the definition of the infimum, there exists a sequence $\{A_i\}$ with $A_i \in \mathcal{A}$ and $E \subset \cup_{i=1}^{\infty} A_i$ such that

$$\sum_{i=1}^{\infty} \mu_0(A_i) - \epsilon \leq \mu^*(E).$$

Let $A = \cup_{i=1}^{\infty} A_i \in \mathcal{A}_\sigma$. Then

$$\mu^*(A) = \mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i) = \sum_{i=1}^{\infty} \mu_0(A_i) \leq \mu^*(E) + \epsilon.$$

(b) First, suppose that E is μ^*-measurable. From part (a), we know that for all $n \in \mathbb{N}$, there exists $B_n \in \mathcal{A}_\sigma$ with $E \subset B_n$ and $\mu^*(B_n) - \mu^*(E) \leq 1/n$. Let

$$B = \bigcap_{n=1}^{\infty} B_n \in \mathcal{A}_{\sigma\delta}.$$

Since E is μ^*-measurable, we have $\mu^*(B_n) = \mu^*(B_n \cap E) + \mu^*(B_n \cap E^c)$, hence

$$\mu^*(B \cap E^c) \leq \mu^*(B_n \cap E^c) = \mu^*(B_n) - \mu(E) \leq 1/n,$$

for every $n \in \mathbb{N}$. Hence $\mu^*(B \backslash E) = 0$.

On the other hand, suppose there exists a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \backslash E) = 0$. From (a), for all $n \in \mathbb{N}$, there exists $A_n \in \mathcal{A}_\sigma$ such that $(B \backslash E) \subset A_n$ and

$$\mu^*(A_n) \leq \mu^*(B \backslash E) + 1/n = 1/n.$$

Therefore, $A = \cap_{n=1}^{\infty} A_n$ is a μ^*-measurable set such that $(B\backslash E) \subset A$ and $\mu^*(A) = 0$. (We know that A is μ^*-measurable since it is in $\mathcal{A}_{\sigma\delta}$ and the set of all μ^*-measurable sets is a σ-algebra.)

By Caratheodory's theorem, we know that μ^* restricted to μ^*-measurable sets is a complete measure. Hence $(B\backslash E)$ is μ^*-measurable. Since $B \in \mathcal{A}_{\sigma\delta}$, we know that B is also μ^*-measurable. Therefore,

$$E = (B^c \cup (B \cap E^c))^c,$$

is in the σ algebra of μ^*-measurable sets. Hence E is μ^*-measurable. Notice that in this direction of the proof, we did not use that $\mu^*(E) < \infty$.

(c) Let μ_0 be σ-finite. Suppose $\mu^*(E) = \infty$ and E is μ^*-measurable. Let

$$X = \bigcup_{i=1}^{\infty} X_i,$$

where $\mu_0(X_i) < \infty$. Denote

$$E_n = (E \cap \bigcup_{i=1}^{n} X_i).$$

Notice that $\mu^*(E_n) < \infty$ and $E = \cup_{n=1}^{\infty} E_n$. Let $\epsilon > 0$. From (a), we know that for all $n \in \mathbb{N}$, there exists $C_n \in \mathcal{A}_\sigma$ such that $E_n \subset C_n$ and

$$\mu^*(C_n \backslash E_n) = \mu^*(C_n) - \mu^*(E_n) \leq \frac{\epsilon}{2^n}.$$

Let $B_\epsilon = \cup_{n=1}^{\infty} C_n \in \mathcal{A}_\sigma$. Since

$$B_\epsilon \cap E^c \subset \bigcup_{n=1}^{\infty} (C_n \cap E_n^c),$$

we have that

$$\mu^*(B_\epsilon \cap E^c) \leq \mu^* \left(\bigcup_{n=1}^{\infty} (C_n \cap E_n^c) \right) \leq \sum_{n=1}^{\infty} \mu^*(C_n \backslash E_n) \leq \epsilon.$$

Now let $B = \cap_{n=1}^{\infty} B_{1/n} \in \mathcal{A}_{\sigma\delta}$. For all $n \in \mathbb{N}$, $\mu^*(B \cap E^c) \leq \mu^*(B_{1/n} \cap E^c) \leq 1/n$, hence $\mu^*(B \cap E^c) = \mu^*(B\backslash E) = 0$.

The reverse direction of (b) did not use the fact that $\mu^*(E) < \infty$. Hence the restriction $\mu^*(E) < \infty$ is superfluous in the case that μ_0 is σ-finite.

Problem 1.21
Let $\mu*$ be an outer measure induced from a premeasure and $\overline{\mu}$ the restriction of μ^* to the μ^*-measurable sets. Then $\overline{\mu}$ is saturated. (Use Exercise 18.)

Solution:
Let $E \subset X$ be such that $E \cap A$ is μ^*-measurable for all A that are μ^*-measurable and are such that $\mu^*(A) < \infty$. We want to show that for all $B \subset X$, we have $\mu^*(B) \geq \mu^*(E \cap B) + \mu^*(E^c \cap B)$. This will complete the proof.

The inequality holds trivially if $\mu^*(B) = \infty$. Suppose $\mu^*(B) < \infty$. Then by 18a, for all $\epsilon > 0$ there exists a μ^*-measurable set A with $B \subset A$ and $\mu^*(A) \leq \mu^*(B) + \epsilon$. Then

$$\begin{aligned}
\mu^*(B) &\geq \mu^*(A) - \epsilon \\
&= \mu^*(A \cap (E \cap A)) + \mu^*(A \cap (E \cap A)^c) - \epsilon \\
&= \mu^*(A \cap E) + \mu^*(A \cap E^c) - \epsilon \\
&\geq \mu^*(B \cap E) + \mu^*(B \cap E^c) - \epsilon.
\end{aligned}$$

Since this holds for all $\epsilon > 0$, we conclude that E is μ^*-measurable.

Problem 1.25
Complete the proof of Theorem 1.19.

Solution:
The proof is done for the case $\mu(E) < \infty$. Suppose $\mu(E) = \infty$. It is clear that (b) and (c) imply (a) since μ is complete on $\mathcal{M}_\mu$. First, we show that (a) implies (b). Denote

$$E_n = E \cap [-n, n],$$

where $n \in \mathbb{N}$. Then $E = \cup_{n=1}^{\infty} E_n$, where $\mu(E_n) < \infty$. Let $\epsilon > 0$. By Theorem 1.19, there exists an open set U_n such that $E_n \subset U_n$, and

$$\mu(U_n) \leq \mu(E_n) + \frac{\epsilon}{2^n}.$$

Since $\mu(U_n) = \mu(U_n \cap E_n^c) + \mu(U_n \cap E_n)$, we have

$$\mu(U_n \backslash E_n) = \mu(U_n) - \mu(E_n) \leq \frac{\epsilon}{2^n}.$$

Let $U = \cup_{n=1}^{\infty} U_n$. Then since $U \cap E^c \subset \cup_{n=1}^{\infty} U_n \cap E_n^c$,

$$\mu(U \cap E^c) \leq \sum_{n=1}^{\infty} \mu(U_n \cap E_n^c) \leq \epsilon.$$

Hence for $\epsilon = 1/k$ with $k \in \mathbb{N}$, there exists an open set U_k such that $\mu(U_k \backslash E) \leq 1/k$. Take $V = \cap_{k=1}^{\infty} U_k$. Then V is a G_δ set such that $\mu(V \backslash E) = 0$.

We will now show that (a) implies (c). Let $E \in \mathcal{M}_\mu$. We know the result holds when $\mu(E) < \infty$. Let $E_n = E \cap [-n, n]$. Then for each E_n, there exists a H_n which is a F_σ set and a N_n with $\mu(N_n) = 0$ such that $E_n = H_n \cup N_n$. Then

$$E = \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} H_n \cup \bigcup_{n=1}^{\infty} N_n = H \cup N,$$

where $H = \cup_{n=1}^{\infty} H_n$ is a F_σ set, and $N = \cup_{n=1}^{\infty} N_n$ is such that

$$\mu(N) \leq \sum_{n=1}^{\infty} \mu(N_n) = 0.$$