Folland: Real Analysis, Chapter 2
Sébastien Picard

Problem 2.3
If $\{f_n\}$ is a sequence of measurable functions on X, then $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

Solution:
Define $h = \limsup f_n$, $g = \liminf f_n$. By Proposition 2.7, h, g are measurable. Let

$$E_\infty = \bigcap_{n=1}^{\infty} g^{-1}(n,\infty) \cap h^{-1}(n,\infty)$$

$$E_{-\infty} = \bigcap_{n=1}^{\infty} g^{-1}(-\infty,-n) \cap h^{-1}(-\infty,-n)$$

.

It is clear that both E_∞ and $E_{-\infty}$ are measurable sets. Next, define

$$w(x) = \begin{cases} 29 & \text{if } g = h = \pm\infty \\ g(x) - h(x) & \text{else} \end{cases}$$

Then w is a measurable function by Exercise 2. Therefore

$$E_1 = (w^{-1}(-\infty,0) \cup w^{-1}(0,\infty))^c$$

is measurable. Hence the set $\{x : \lim f_n \text{ exists}\}$ is measurable since it is equal to

$$\{x \in X : g(x) = h(x)\} = E_1 \cup E_\infty \cup E_{-\infty}.$$

Problem 2.9
Let $f : [0,1] \to [0,1]$ be the Cantor function (§1.5), and let $g(x) = f(x) + x$.

a. *g is a bijection from $[0,1]$ to $[0,2]$ and $h = g^{-1}$ is continuous from $[0,2]$ to $[0,1]$.*

b. *If C is the Cantor set, $m(g(C)) = 1$.*

c. *By Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.*

d. *There exists a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.*

Solution:
(a) We know that f is an increasing function, and therefore $f(x)+x$ is a strictly increasing function. Also, since f is continuous, then $f(x) + x$ is continuous. Since $g(0) = 0$ and $g(1) = 2$, and g is continuous and strictly increasing on $[0,1]$, then g is a bijection from $[0,1]$ to $[0,2]$. It follows that g^{-1} exists.

Since g is continuous map from a compact set to a Hausdorff space, then g^{-1} is continuous: indeed, $g(K)$ is compact for any compact set K, and compacts sets are closed in a Hausdorff space, so $(g^{-1})^{-1}(C)$ is closed for any closed set C in the domain of g.

(b) Let

$$[0,1] \cap C^c = \bigsqcup_{n=1}^{\infty} E_n$$

where E_n are countably many disjoint intervals. Since f is constant on a given E_n, then $g(E_n)$ is a translate of E_n, so $m(g(E_n)) = m(E_n)$. Therefore

$$m(g([0,1] \cap C^c)) = m(\bigsqcup_{m=1}^{\infty} g(E_n)) = m(\bigsqcup_{m=1}^{\infty} E_n) = m([0,1] \cap C^c) = 1$$

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Since $2 = m([0,2]) = m(g(C)) + m(g([0,1] \cap C^c))$, we have $m(g(C)) = 1$.

(c) Suppose B is a Borel set. Since g^{-1} is continuous, $(g^{-1})^{-1}(B) = A$ is Borel measurable. But A is not even Lebesgue measurable, which is a contradiction.

However, $B \subset C$. Since C is a null set, B is Lebesgue measurable by completeness of Lebesgue measure.

(d) Define G and F as follows:

$$G(x) = \begin{cases} g^{-1}(x) & \text{if } x \in [0,2] \\ -x & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 2 \end{cases}$$

$$F(x) = \begin{cases} x & \text{if } x \in B \\ -69 & \text{else} \end{cases}$$

G is continuous by the Pasting Lemma from elementary topology. F is measurable since $F^{-1}(a, \infty)$ is either empty, the whole real line, or a subset of B (which is measurable since B has measure zero).

$F \circ G$ is not Lebesgue measurable since $G^{-1}(F^{-1}(-1, \infty)) = G^{-1}(B) = A$.

Problem 2.14

If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g d\lambda = \int f g d\mu$. (First suppose that g is simple.)

Solution:

We will show that λ is a measure on $\mathcal{M}$. It is clear that $\lambda(\emptyset) = 0$. Let $\{A_k\}$ be a countable sequence of disjoint sets, and define $A = \cup_{k=1}^{\infty} A_k$.

$$\lambda(\cup A_k) = \int_A f d\mu = \int_A (\sum_{k=1}^{\infty} \chi_{A_k} f) d\mu = \sum_{k=1}^{\infty} \int_A \chi_{A_k} f d\mu = \sum_{k=1}^{\infty} \lambda(A_k).$$

The summation can be taken out from the integral by Theorem 2.15.

Let $g \in L^+$ be a simple function. Then $g = \sum_{i=1}^n a_i \chi_{A_i}$ where A_i are measurable sets. Therefore

$$\int g d\lambda = \sum_{i=1}^n a_i \lambda(A_i) = \sum_{i=1}^n a_i \int f \chi_{A_i} d\mu = \int \sum_{i=1}^n a_i \chi_{A_i} f d\mu = \int f g d\mu.$$

Now take any $g \in L^+$. By Theorem 2.10 there exists a sequence $\{\phi_n\}$ of simple functions that converges to g pointwise such that $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq g$. Since $\{\phi_n\}$ and $\{f\phi_n\}$ are monotone increasing, we can apply the monotone convergence theorem twice to obtain

$$\int g d\lambda = \int \lim \phi_n d\lambda = \lim \int \phi_n d\lambda = \lim \int \phi_n f d\mu = \int f g d\mu.$$

Problem 2.16

If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Solution:

Let $f \in L^+$ such that $\int f < \infty$. Let $\epsilon > 0$. By definition of $\int f$, there exists a simple function $\phi = \sum_{i=1}^n a_i \chi_{E_i}$ such that $0 \leq \phi \leq f$ and

$$\int f - \epsilon < \int \phi.$$

Let $E = \cup_{i=1}^n E_i$. Since the E_i are in $\mathcal{M}$, then $E \in \mathcal{M}$. Also, $\int \phi \leq \int f < \infty$, so for each E_i we have $\mu(E_i) < \infty$ and therefore $\mu(E) < \infty$. Hence we have

$$\int f - \epsilon < \int_E \phi \leq \int_E f.$$

Problem 2.20

(A generalized Dominated Convergence Theorem) If f_n, g_n, f, $g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem.)

Solution:

Since $f_n + g_n \geq 0$, we can apply Fatou's lemma:

$$\int f + g = \int \lim(f_n + g_n) \leq \liminf \int f_n + g_n = \int g + \liminf \int f_n.$$

The same process can be repeated for $g - f_n$:

$$\int g - f = \int \lim(g_n - f_n) \leq \liminf \int g_n - f_n = \int g - \limsup \int f_n.$$

Since

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n$$

we can conclude that $\int f_n \to \int f$.

Problem 2.21
Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. (Use Exercise 20.)

Solution:
Suppose $\int |f_n - f| \to 0$. Then

$$\limsup \int |f_n| \leq \limsup \int |f_n - f| + |f| = \int |f|.$$

On the other hand, we can apply Fatou's lemma to obtain

$$\int |f| \leq \liminf \int |f_n|.$$

Therefore

$$\limsup \int |f_n| \leq \int |f| \leq \liminf \int |f_n|$$

and hence $\int |f_n| \to \int |f|$.

Next we prove the converse: suppose $\int |f_n| \to \int |f|$. Define $g_n = |f_n| + |f|$. Then $|f_n - f| \leq g_n$ and $\int g_n \to 2 \int |f| \in L^1$. By applying the generalized Dominated Convergence Theorem (Problem 20) we obtain

$$\lim \int |f_n - f| = \int \lim |f_n - f| = 0.$$

Problem 2.25
Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.

a. *$g \in L^1(m)$, and in particular $g < \infty$ a.e.*

b. *g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.*

c. *$g^2 < \infty$, but g^2 is not integrable on any interval.*

Solution:
(a) We need to show $\int |g| < \infty$. Using Theorem 2.15 and the change of variables $z = x - r_n$, we can compute

$$\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}2^{-n}f(x-r_n)dx = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}2^{-n}f(x-r_n)dx$$
$$= \sum_{n=1}^{\infty}2^{-n}\int_0^1 f(z)dz$$
$$= \sum_{n=1}^{\infty}2^{-n}\int_0^1 z^{-1/2}dz$$
$$= \sum_{n=1}^{\infty}2^{1-n}$$
$$= 2(\sum_{n=0}^{\infty}2^{-n}-1) = 2 < \infty.$$

(b) Let $x \in \mathbb{R}, M > 0, \delta > 0$. It will be shown that there exists a $y \in (x-\delta, x+\delta)$ such that $g(y) > M$. This will prove that g is unbounded on each interval and that g is discontinuous at every point.

There exists a rational number $r_n \in (x-\delta, x+\delta)$. Furthermore, there exists a $y \in (x-\delta, x+\delta)$ such that

$$0 < y - r_n < \frac{2^{-2n}}{4M^2}.$$

Then

$$g(y) \geq 2^{-n}f(y-r_n) \geq 2^{-n}(2^n(2M)) = 2M > M.$$

This proof does not fail after redefining g on any Lebesgue null set, since one can still find an irrational y with the desired properties.

(c) Since $g < \infty$ a.e., it follows that $g^2 < \infty$ a.e. However,

$$g^2 \geq \int\sum_{n=1}^{\infty}2^{-2n}f^2(x-r_n)dx = \sum_{n=1}^{\infty}2^{-2n}\int_0^1 f^2(z)dz \geq \frac{1}{4}\int_0^1\frac{dz}{z} > \infty.$$

Problem 2.27

Let $f_n(x) = ae^{-nax} - be^{-nbx}$ *where* $0 < a < b$.

a. $\sum_1^\infty\int_0^\infty|f_n(x)|dx = \infty$.

b. $\sum_1^\infty\int_0^\infty f_n(x)dx = 0$.

c. $\sum_1^\infty f_n \in L^1([0,\infty), m)$, *and* $\int_0^\infty\sum_1^\infty f_n(x)dx = \log(b/a)$.

Solution:

(a) Since f_n is the difference of two exponential functions, we can find a point $c \in \mathbb{R}$ such that $f_n < 0$ on $(0, c)$ and $f_n > 0$ on (c, ∞). In order to find this point c, we solve

$$\begin{aligned} be^{-nbc} &= ae^{-nac} \\ \log(b/a) &= nc(b-a) \\ c &= \log(b/a)\frac{1}{n(b-a)} \end{aligned}$$

We can now split up the integral in order to integrate $|f_n|$:

$$\begin{aligned} \int_0^\infty |f_n| &= -\int_o^c \Big(ae^{-nax} - be^{-nbx}\Big)dx + \int_c^\infty \Big(ae^{-nax} - be^{-nbx}\Big)dx \\ &= \Big(\frac{e^{-nax}}{n} - \frac{e^{-nbx}}{n}\Big)\Big|_0^c + \Big(\frac{e^{-nbx}}{n} - \frac{e^{-nax}}{n}\Big)\Big|_c^\infty \\ &= \frac{2}{n}\Big(e^{-nac} - e^{-nbc}\Big) \\ &= \frac{2}{n}\Big(e^{-\frac{a}{b-a}\log(b/a)} - e^{-\frac{b}{b-a}\log(b/a)}\Big). \end{aligned}$$

Therefore $\int_0^\infty |f_n|$ is proportional to $(1/n)$, hence $\sum_1^\infty \int_0^\infty |f_n| = \infty$.

(b)

$$\begin{aligned} \int_0^\infty f_n(x)dx &= \int_o^\infty ae^{-nax}dx - \int_0^\infty be^{-nbx}dx \\ &= \frac{e^{-nax}}{-n}\Big|_0^\infty + \frac{e^{-nbx}}{n}\Big|_0^\infty \\ &= \frac{1}{n}(-1+1) = 0. \end{aligned}$$

Therefore $\sum_1^\infty \int_0^\infty f_n(x)dx = 0$.

(c)

$$\begin{aligned} \sum_{n=1}^\infty f_n &= \sum_{n=1}^\infty \Big(ae^{-nax} - be^{-nbx}\Big) \\ &= a\Big(\sum_{n=0}^\infty e^{-nax} - 1\Big) - b\Big(\sum_{n=0}^\infty e^{-nbx} - 1\Big) \\ &= a\Big(\frac{1}{1-e^{-ax}} - 1\Big) - b\Big(\frac{1}{1-e^{-bx}} - 1\Big) \\ &= \frac{a}{1-e^{-ax}} - \frac{b}{1-e^{-bx}} + (b-a). \end{aligned}$$

We can see that $\sum_{n=1}^{\infty} f_n$ is positive on $(0,\infty)$, so to show that $\sum_{n=1}^{\infty} f_n \in L^1$ we must compute $\int_0^\infty \sum_{n=1}^{\infty} f_n$:

$$\begin{aligned}
\int_o^\infty \sum_{n=1}^{\infty} f_n &= \int_0^\infty \Big(\frac{a}{1-e^{-ax}} - \frac{b}{1-e^{-bx}} + (b-a)\Big)dx \\
&= \Big(\log(1-e^{ax}) - \log(1-e^{bx}) + (b-a)x\Big)\Big|_0^\infty \\
&= \log\Big(\frac{(1-e^{ax})(e^{(b-a)x})}{1-e^{bx}}\Big)\Big|_0^\infty \\
&= \lim_{z\to 0}\lim_{x\to\infty} \log\Big(\frac{(1-e^{ax})(1-e^{bz})e^{bx}e^{-ax}}{(1-e^{bx})(1-e^{az})e^{bz}e^{-az}}\Big) \\
&= \lim_{z\to 0}\lim_{x\to\infty} \log\Big(\frac{(1-e^{bz}-e^{ax}+e^{ax+bz})e^{bx}e^{az}}{(1-e^{az}-e^{bx}+e^{bx+az})e^{bz}e^{ax}}\Big) \\
&= \lim_{z\to 0}\lim_{x\to\infty} \log\Big(\frac{e^{ax}(e^{-ax}-e^{bz-ax}-1+e^{bz})e^{bx}e^{az}}{e^{bx}(e^{-bx}-e^{az-bx}-1+e^{az})e^{bz}e^{ax}}\Big) \\
&= \lim_{z\to 0}\lim_{x\to\infty} \log\Big(\frac{(e^{-ax}-e^{bz-ax}-1+e^{bz})e^{az}}{(e^{-bx}-e^{az-bx}-1+e^{az})e^{bz}}\Big) \\
&= \lim_{z\to 0} \log\Big(\frac{(e^{bz}-1)e^{az}}{(e^{az}-1)e^{bz}}\Big) \\
&= \lim_{z\to 0} \log\Big(\frac{1-e^{-bz}}{1-e^{-az}}\Big) \\
&= \log(b/a).
\end{aligned}$$

The last step follows from l'Hopital's rule:

$$\lim_{z\to 0} \frac{1-e^{-bz}}{1-e^{-az}} = \lim_{z\to 0} \frac{be^{-bz}}{ae^{-az}} = \frac{b}{a}.$$

Problem 2.28

Compute the following limits and justify the calculations:

a. $\lim_{n\to\infty} \int_0^\infty (1+(x/n))^{-n}\sin(x/n)dx$.

b. $\lim_{n\to\infty} \int_0^1 (1+nx^2)(1+x^2)^{-n}dx$.

c. $\lim_{n\to\infty} \int_0^\infty n\sin(x/n)[x(1+x^2)]^{-1}dx$.

d. $\lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1}dx$. *(The answer depends on whether $a>0$, $a=0$, or $a<0$. How does this accord with the various convergence theorems?*

Solution:

(a) Denote

$$f_n = \frac{\sin(x/n)}{(1+(x/n))^n}.$$

First, notice that by the binomial theorem, when $n \geq 2$, for all $x \in [0,\infty)$ we have

$$(1+(x/n))^n \geq 1+x+\binom{n}{2}\frac{x^2}{n^2} \geq 1+x+\frac{x^2}{4}.$$

Thus we can bound f_n on $[0,\infty)$ by

$$|f_n| = \left|\frac{\sin(x/n)}{(1+(x/n))^n}\right| \leq \frac{1}{(1+(x/n))^n} \leq \frac{1}{1+x+x^2/4}.$$

A rough computation shows that the following integral is bounded:

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x+x^2/4} &= \int_0^1 \frac{dx}{1+x+x^2/4} + \int_1^\infty \frac{dx}{1+x+x^2/4} \\ &\leq \int_0^1 \frac{dx}{1+0+0} + \int_1^\infty \frac{dx}{0+0+x^2/4} \\ &\leq 1+4 < \infty.\end{aligned}$$

By the Dominated Convergence Theorem, we can throw the limit under the integral.

$$\lim_{n\to\infty}\left|\int_0^\infty f_n dx\right| \leq \int_0^\infty \lim_{n\to\infty}|f_n|dx = \int_0^\infty \lim_{n\to\infty}\left|\frac{\sin(x/n)}{(1+(x/n))^n}\right|dx = \int_0^\infty \sin(0)e^{-x}dx = 0.$$

(b) Denote

$$f_n = \frac{1+nx^2}{(1+x^2)^n}.$$

By the binomial theorem, we know

$$(1+x^2)^n \geq 1+nx^2.$$

Therefore $f_n \leq 1$, and $\int_0^1 1 < \infty$, so by the Dominated Convergence Theorem we can conclude

$$\lim_{n\to\infty}\int_0^1 f_n dx = \int_0^1 \left(\lim_{n\to\infty}\frac{1+nx^2}{(1+x^2)^n}\right)dx = 0.$$

(c) Denote

$$f_n = \frac{n\sin(x/n)}{x(1+x^2)}.$$

Since $|\sin(x/n)| \leq (x/n)$ for $x>0$, we can bound f_n on $(0,\infty)$ by

$$\left|\frac{n\sin(x/n)}{x(1+x^2)}\right| \leq \frac{1}{1+x^2}.$$

Next we verify that

$$\int_0^\infty \frac{dx}{1+x^2} = \arctan(x)\Big|_0^\infty = \pi/2 < \infty.$$

By the Dominated Convergence Theorem we can conclude

$$\lim_{n\to\infty}\int_0^\infty f_n dx = \int_0^\infty \Big(\lim_{n\to\infty}\frac{\sin(x/n)}{x/n}(1+x^2)^{-1}\Big)dx = \int_0^\infty \frac{dx}{1+x^2} = \pi/2.$$

(d) In this case we can evaluate the integral directly:

$$\lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}dx = \lim_{n\to\infty}\int_{na}^\infty \frac{dy}{1+y^2} = \lim_{n\to\infty}\arctan(y)\Big|_{na}^\infty = \pi/2 - \lim_{n\to\infty}\arctan(na).$$

Hence

$$\lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}dx = \begin{cases} 0 & a>0 \\ \pi/2 & a=0 \\ \pi & a<0 \end{cases}$$

This agrees with the fact that one cannot apply the Dominated Convergence Theorem unless $a>0$ since there is no way to bound $f_n(0)$.

Problem 2.32

Suppose $\mu(X)<\infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}d\mu.$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

Solution:

We will first show that ρ is a metric on the space of measurable functions if we identify functions that are equal a.e.

It is clear that $\rho(f,g)=\rho(g,f)$ and that $\rho(f,g)\geq 0$. We can also see that

$$\begin{aligned}\rho(f,g)=0 &\iff \int \frac{|f-g|}{1+|f-g|}d\mu = 0 \\ &\iff \frac{|f-g|}{1+|f-g|} = 0 \text{ a.e. (by Proposition 2.16)} \\ &\iff f=g \text{ a.e.}\end{aligned}$$

It only remains to show the triangle inequality. Let $x,y,z\in X$. First, suppose $|x-z|\leq|x-y|$ and $|z-y|\leq|x-y|$. Then we have

$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-y|} + \frac{|z-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}.$$

On the other hand, suppose $|x-z| \geq |x-y|$. Then

$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|z-y|}{1+|z-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}.$$

The above argument can be repeated when $|z-y| \geq |x-y|$. Hence for all $x, y, z \in X$,

$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}.$$

Using basic properties of the integral (Proposition 2.13), we can conclude that $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$ for any measurable functions f, g, h. This completes the proof that ρ is a metric.

Suppose $f_n \to f$ with respect to ρ. Let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| \geq \epsilon\}$.

$$\int \frac{|f_n - f|}{1+|f_n - f|} d\mu \geq \int_{E_{n,\epsilon}} \frac{|f_n - f|}{1+|f_n - f|} d\mu \geq \frac{\epsilon}{1+\epsilon} \mu(E_{n,\epsilon}).$$

It follows that

$$\mu(E_{n,\epsilon}) \leq \frac{1+\epsilon}{\epsilon} \int \frac{|f-g|}{1+|f-g|} d\mu \to 0.$$

Conversely, suppose $f_n \to f$ in measure. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all integers $n > N$, we have

$$\mu\left(\left\{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2\mu(X)}\right\}\right) < \frac{\epsilon}{2}.$$

Let $A = \{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2\mu(X)}\}$. Then the following holds for all $n > N$:

$$\begin{aligned}
\rho(f_n, f) = \int \frac{|f-g|}{1+|f-g|} d\mu &= \int_A \frac{|f-g|}{1+|f-g|} d\mu + \int_{X \backslash A} \frac{|f-g|}{1+|f-g|} d\mu \\
&\leq \mu(A) + \frac{\epsilon}{2\mu(X)} \mu(X \backslash A) \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Problem 2.40

In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leq g$ for all n, where $g \in L^1(\mu)$."

Solution:

Suppose $f_1, f_2, \ldots$ and f are measurable complex-valued functions on X such that $f_n \to f$ a.e. and $|f_n| \leq g$ for all n, where $g \in L^1(\mu)$. We will follow the proof of Theorem 2.33 and make some minor adjustments.

Without loss of generality, assume that $f_n \to f$ everywhere on X. For $k, n \in \mathbb{N}$, let

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x : |f_m(x) - f(x)| \geq k^{-1}\}.$$

For fixed k, $E_n(k)$ decreases as n increases and $\cap_{n=1}^{\infty} E_n(k) = \emptyset$. To apply continuity of measure from above, we need $\mu(E_1) < \infty$. Since $|f_n - f| \leq 2|g|$, we observe that

$$E_1(k) \subseteq A(k) := \{x : 2|g(x)| \geq k^{-1}\}.$$

We can use the fact that

$$\infty > 2\int_X |g| \geq \int_{A(k)} 2|g| \geq k^{-1}\mu(A(k)),$$

in order to conclude

$$\mu(E_1(k)) \leq \mu(A(k)) < \infty.$$

Therefore, by continuity of measure from above, $\mu(E_n(k)) \to 0$ as $n \to \infty$. Given $\epsilon > 0$ and $k \in \mathbb{N}$, there exists a positive integer n_k such that $\mu(E_{n_k}(k)) < \epsilon 2^{-k}$.

If we define $E = \cup_{k=1}^{\infty} E_{n_k}(k)$, then $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

Problem 2.49

Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

a. *If $E \in \mathcal{M} \times \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.*

b. *If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x \, d\nu = \int f^y \, d\mu = 0$ for a.e. x and y. (Here the completeness of μ and ν is needed.)*

Solution:

(a) Suppose $E \in M \times N$ and $\mu \times \nu(E) = 0$. Define $f = \chi_E$. Then $f_x = \chi_{E_x}$ and $f^y = \chi_{E^y}$. Apply Fubini's theorem:

$$0 = \int f d(\mu \times \nu) = \int \Big(\int f_x d\nu(y)\Big) d\mu(x) = \int \Big(\int f^y d\mu(x)\Big) d\nu(y).$$

It follows that $\int \chi_{E_x} d\nu = 0$ μ-a.e. and $\nu(E_x) = 0$ μ-a.e., and similarly $\int \chi_{E^y} d\mu = 0$ ν-a.e. and $\mu(E^y) = 0$ ν-a.e..

(b) Suppose f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e.. Define

$$A = \{(x, y) \in M \times N : f(x, y) \neq 0\}.$$

Then $A \subseteq E$ for some $E \in M \otimes N$ such that $\mu \times \nu(E) = 0$. By part (a), $\nu(E_x) = 0$ and $\mu(E^y) = 0$ for a.e. x and y. Since $A_x \subset E_x$ and $A^y \subset E^y$, we have $\nu(A_x) = 0$ and $\mu(A^y) = 0$. Therefore $\int |f_x| d\mu = \int \chi_{A_x} |f_x| d\mu = 0$ for μ-a.e. x and $\int |f^y| d\nu = \int \chi_{A^y} |f^y| d\nu = 0$ for ν-a.e. y.

We now prove Theorem 2.39. Suppose f is $\mathcal{L}$-measurable and either (a) $f \geq 0$ or (b) $f \in L^1(\lambda)$. By Proposition 2.12, there exists a $M \otimes N$-measurable function g such that $f = g$ λ-almost everywhere. By Proposition 2.34, g_x is N-measurable and g^y is M-measurable. Define $h = g - f$. Then $h = 0$ λ-a.e.. By lemma (b), h_x is N-measurable for a.e. x and h^y is M-measurable for almost every y, hence f_x is N-measurable for a.e. x and f^y is M-measurable for almost every y

In case (b), $g \in L^1(\mu \times \nu)$, so by lemma (b), $h_x = g_x - f_x$ is integrable for a.e. x. By Fubini's theorem, g_x is integrable for a.e. x, hence f_x is integrable for a.e. x. Similarly f^y is integrable for a.e. y.

By applying lemma (b) on the function h, we can see that for a.e. x we have $\int (g_x - f_x) d\nu = 0$ hence $\int g_x d\nu = \int f_x d\nu$. Similarly for a.e. y we have $\int g^y d\mu = \int f^y d\mu$. In case (a), $g \in L^+(M \times N)$, so by Tonelli's theorem $x \to \int g_x d\nu = \int f_x d\nu$ is measurable, and $y \to \int g^y d\nu = \int f^y d\nu$ is measurable. In case (b), $g \in L^1(M \times N)$, so by Fubini's theorem $x \to \int g_x d\nu = \int f_x d\nu$ is integrable, and $y \to \int g^y d\nu = \int f^y d\nu$ is integrable.

Since $\int g d\lambda = \int f d\lambda$, after applying Tonelli's theorem (case a) or Fubini's theorem (case b) and using the fact that $\int g^y d\nu = \int f^y d\nu$ and $\int g_x d\mu = \int f_x d\mu$ for almost every x and y, we obtain

$$\int f d\lambda = \int \int f(x,y) d\mu d\nu = \int \int f(x,y) d\nu d\mu.$$

Problem 2.55

Let $E = [0,1] \times [0,1]$. Investigate the existence and equality of $\int_E f dm^2$, $\int_0^1 \int_0^1 f(x,y)\ dx\ dy$, and $\int_0^1 \int_0^1 f(x,y)\ dy\ dx$ for the following f.

a. $f(x,y) = (x^2 - y^2)(x^2 + y^2)^{-2}$.

b. $f(x,y) = (1 - xy)^{-a}$ $(a > 0)$.

c. $f(x,y) = (x - 1/2)^{-3}$ *if* $0 < y < |x - 1/2|$, $f(x,y) = 0$ *otherwise.*

Solution:

(a) First, we evaluate

$$\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx.$$

Using the substitution $x = y \tan\theta$, $dx = y \sec^2\theta d\theta$, we obtain

$$\begin{aligned}
\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx &= \int_0^{\arctan(1/y)} \frac{(\tan^2\theta - 1)y^3\sec^2\theta d\theta}{(y^2(\tan^2\theta+1))^2} \\
&= \int_0^{\arctan(1/y)} \frac{(\tan^2\theta-1)d\theta}{y\sec^2\theta} \\
&= \frac{1}{y}\int_0^{\arctan(1/y)} (\sin^2\theta - \cos^2\theta)d\theta \\
&= \frac{1}{y}\int_0^{\arctan(1/y)} (1-2\cos^2\theta)d\theta \\
&= \frac{1}{y}\int_0^{\arctan(1/y)} \cos 2\theta d\theta \\
&= \frac{-1}{2y}\sin(2\arctan(1/y)) \\
&= \frac{-1}{y}\sin(\arctan(1/y))\cos(\arctan(1/y)) \\
&= \frac{-1}{y}\frac{\sin(\arctan(1/y))}{\cos(\arctan(1/y))}\frac{\cos(\arctan(1/y))}{\frac{1}{\cos(\arctan(1/y))}} \\
&= \frac{-1}{y}\frac{\tan(\arctan(1/y))}{1+\tan^2(\arctan(1/y))} \\
&= \frac{-1}{1+y^2}
\end{aligned}$$

Therefore,

$$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dxdy = \int_0^1 \frac{-dy}{1+y^2} = -\pi/4.$$

We observe that

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx = -\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}dx = -\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy.$$

Hence

$$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dydx = \int_0^1 \frac{dx}{1+x^2} = \pi/4.$$

By Fubini's theorem, $\int_E f dm^2$ is not defined.

(b) Since f is non-negative on $[0,1]\times[0,1]$, $f \in L^+(E)$, so by Tonelli's theorem $\int_E f dm^2 = \int_0^1\int_0^1 f(x,y)dxdy = \int_0^1\int_0^1 f(x,y)dydx$. The integral may be infinite for some values of a... I haven't had time to do this computation yet.

(c) First, we compute

$$\int_0^1 \int_0^1 f(x,y)dydx = \int_0^1 \int_0^{|x-0.5|} (x-\frac{1}{2})^{-3}dydx = \int_0^1 \frac{(x-\frac{1}{2})dx}{|x-\frac{1}{2}|^3}.$$

The function $\frac{(x-\frac{1}{2})dx}{|x-\frac{1}{2}|^3}$ is not integrable on $[0,1]$:

$$\int_0^1 \frac{|x-\frac{1}{2}|dx}{|x-\frac{1}{2}|^3} = \int_0^1 \frac{dx}{|x-\frac{1}{2}|^2} = \infty.$$

Therefore, the integral $\int_0^1\int_0^1 f(x,y)dydx$ does not exist, and hence $\int_E f dm^2$ does not exist. However, $\int_0^1\int_0^1 f(x,y)dxdy = 0$:

$$\begin{aligned}\int_0^1 f(x,y)dx &= \int_0^1 \chi_{\{y<|x-1/2|\}}(x-\frac{1}{2})^{-3}dx \\ &= \int_{-1/2}^{1/2} \chi_{\{|z|>y\}}\frac{dz}{z^3} \\ &= \int_{-1/2}^{-y} \frac{dz}{z^3} + \int_y^{1/2} \frac{dz}{z^3} \\ &= -2 + \frac{1}{2y^2} - \frac{1}{2y^2} + 2 = 0.\end{aligned}$$

Problem 2.57

Show that $\int_0^\infty e^{-sx}x^{-1}\sin x\ dx = \arctan(s^{-1})$ *for* $s>0$ *by integrating* $e^{-sxy}\sin x$ *with respect to* x *and* y*. (It may be useful to recall that* $\tan(\pi/2-\theta) = (\tan\theta)^{-1}$*. Cf. Exercise* 31*d.)*

Solution:

We will investigate the integral

$$\int_0^\infty \int_1^\infty e^{-sxy}\sin x dydx.$$

We want to apply Fubini's theorem, so first we verify that

$$\int_0^\infty \int_1^\infty |e^{-sxy}\sin x|dydx \le \int_0^\infty \int_1^\infty e^{-sxy}xdydx = \int_0^\infty \frac{e^{-sx}}{s}dx < \infty.$$

Next, we observe that

$$\int_0^\infty \int_1^\infty e^{-sxy}\sin xdydx = \frac{1}{s}\int_0^\infty e^{-sx}x^{-1}\sin xdx.$$

This integral can be computed by switching the order of integration. The first step is to use integration by parts to compute

$$
\begin{aligned}
I = \int_0^\infty e^{-sxy} \sin x dx &= \int_0^\infty (-sy) e^{-sxy} \cos x dx - e^{sxy} \cos x \Big|_0^\infty \\
&= \int_0^\infty (-sy) e^{-sxy} \cos x dx + 1 \\
&= \int_0^\infty (-s^2 y^2) e^{-sxy} \sin x dx - sye^{-syx} \sin x \Big|_0^\infty + 1 \\
&= \int_0^\infty (-s^2 y^2) e^{-sxy} \sin x dx + 1.
\end{aligned}
$$

$$I = -s^2 y^2 I + 1.$$

Therefore

$$I = \frac{1}{1 + s^2 y^2}.$$

Thus

$$\int_1^\infty \int_0^\infty e^{-sxy} \sin x dx dy = \int_1^\infty \frac{dy}{1 + s^2 y^2} = \frac{1}{s} \int_s^\infty \frac{dz}{1 + z^2} = \frac{1}{s} \Big(\frac{\pi}{2} - \arctan(s) \Big).$$

It is given that

$$\tan(\frac{\pi}{2} - \theta) = (\tan \theta)^{-1}$$

which implies

$$\tan \left(\frac{\pi}{2} - \arctan(s) \right) = \frac{1}{s}.$$

Therefore

$$\frac{\pi}{2} - \arctan(s) = \arctan(s^{-1}).$$

By Fubini's theorem, we conclude

$$
\begin{aligned}
\frac{1}{s} \int_0^\infty e^{-sx} x^{-1} \sin x dx &= \frac{1}{s} \arctan(s^{-1}). \\
\int_0^\infty e^{-sx} x^{-1} \sin x dx &= \arctan(s^{-1}).
\end{aligned}
$$

Problem 2.58

Show that $\int e^{-sx} x^{-1} \sin^2 x \ dx = \frac{1}{4} \log(1 + 4s^{-2})$ *for* $s > 0$ *by integrating* $e^{-sx} \sin 2xy$ *with respect to* x *and* y*.*

Solution:

We will investigate the integral

$$\int_0^\infty \int_0^1 e^{-sx}\sin(2xy)dydx.$$

We want to apply Fubini's theorem, so first we verify that

$$\int_0^\infty \int_0^1 |e^{-sx}\sin(2xy)|dydx \le \int_0^\infty \int_0^1 e^{-sxy}2xydydx = \int_0^\infty e^{-sx}xdx < \infty.$$

Next, we observe that

$$\int_0^\infty \int_0^1 e^{-sx}\sin(2xy)dydx = \int_0^\infty \frac{e^{-sx}}{x}\Big(\frac{1}{2}-\frac{1}{2}\cos 2x\Big) = \int_0^\infty \frac{e^{-sx}\sin^2 xdx}{x}.$$

This integral can be computed by switching the order of integration. The first step is to use integration by parts to compute

$$\begin{aligned} I = \int_0^\infty e^{-sx}\sin(2xy)dx &= \int_0^\infty \frac{e^{-sx}}{s}2y\cos(2yx)dx - \frac{e^{-sx}}{s}\sin(2yx)\Big|_0^\infty \\ &= \int_0^\infty \frac{e^{-sx}}{s}2y\cos(2yx)dx \\ &= \int_0^\infty \frac{-(2y)^2e^{-sx}}{s^2}\sin(2yx)dx - \frac{2y}{s^2}e^{-sx}\cos(2yx)\Big|_0^\infty \\ &= \int_0^\infty \frac{-(2y)^2e^{-sx}}{s^2}\sin(2yx)dx + \frac{2y}{s^2}. \end{aligned}$$

$$I = \frac{-4y^2 I}{s^2} + \frac{2y}{s^2}.$$

$$I = \frac{2y}{s^2+4y^2}.$$

Therefore

$$\int_0^1 \int_0^\infty e^{-sx}\sin(2xy)dxdy = \int_0^1 \frac{2y}{s^2+4y^2}dy = \frac{1}{4}\log(s^2+4) - \frac{1}{4}\log(s^2) = \frac{1}{4}\log(1+4s^{-2}).$$

By Fubini's theorem, this proves that

$$\int_0^\infty \frac{e^{-sx}\sin^2 xdx}{x} = \frac{1}{4}\log(1+4s^{-2}).$$

Problem 2.60

$\Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ *for* $x, y > 0$. *(Recall that* Γ *was defined in §2.3. Write* $\Gamma(x)\Gamma(y)$ *as a double integral and use the argument of the exponential as a new variable of integration.)*

Solution:

By definition, we have

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty t^{x-1}s^{y-1}e^{-t-s}dsdt.$$

We perform the change of variables $s = u - uv$, $t = uv$. The Jacobian is

$$\frac{\partial(s,t)}{\partial(u,v)} = \frac{\partial s}{\partial u}\frac{\partial t}{\partial v} - \frac{\partial s}{\partial v}\frac{\partial t}{\partial u} = (1-v)u - (-uv) = u.$$

The change of variables formula for multiple integrals yields

$$\Gamma(x)\Gamma(y) = \int_0^1 \int_0^\infty (uv)^{x-1}(u-uv)^{y-1}e^{-u}ududv = \int_0^1 \int_0^\infty v^{x-1}(1-v)^{y-1}e^{-u}u^{x+y-1}dudv.$$

By definition of $\Gamma(x+y)$ and Fubini's theorem, this can be rewritten as

$$\Gamma(x)\Gamma(y) = \Big(\int_0^1 v^{x-1}(1-v)^{y-1}dv\Big)\Gamma(x+y).$$

This proves that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Problem 2.63

The technique used to prove Proposition 2.54 *can also be used to integrate any polynomial over* S^{n-1}. *In fact, suppose* $f(x) = \prod_1^n x_j^{\alpha_j}$ *(*$\alpha_j \in \mathbb{N} \cup \{0\}$*) is a monomial. Then* $\int f d\sigma = 0$ *if any* α_j *is odd, and if all* α_j*'s are even,*

$$\int f d\sigma = \frac{2\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1+\cdots+\beta_n)}, \text{ where } \beta_j = \frac{\alpha_j+1}{2}.$$

Solution:

By Theorem 2.49, we know that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\prod_{j=1}^n x_j^{\alpha_j}dx = \int_0^\infty \int_{S^{n-1}} e^{-r^2}\prod_{j=1}^n \Big(\frac{x_j}{|x_j|}\Big)^{\alpha_j} r^{\sum\alpha_j}r^{n-1}d\sigma dr.$$

First we compute the right-hand side:

$$\begin{aligned}\int_0^\infty \int_{S^{n-1}} e^{-r^2}\prod_{j=1}^n \Big(\frac{x_j}{|x_j|}\Big)^{\alpha_j} r^{\sum\alpha_j}r^{n-1}d\sigma dr &= \Bigg(\int_{S^{n-1}} f d\sigma\Bigg)\Bigg(\int_0^\infty e^{-r^2}r^{n-1+\sum\alpha_j}dr\Bigg)\\ &= \Bigg(\int_{S^{n-1}} f d\sigma\Bigg)\Bigg(\int_0^\infty \frac{e^{-s}}{2}s^{\sum\frac{\alpha_j}{2}}s^{\frac{n-1}{2}}s^{-\frac{1}{2}}ds\Bigg)\\ &= \Bigg(\int_{S^{n-1}} f d\sigma\Bigg)\frac{1}{2}\Gamma(\sum\frac{\alpha_j+1}{2}).\end{aligned}$$

If any α_i is odd, then $\int_{-\infty}^{\infty} e^{-x_i^2} x_i^{\alpha_i} dx_i = 0$ by symmetry, hence then left-hand side is zero:

$$\int_{\mathbb{R}^n} e^{-|x|^2} \prod_{j=1}^{n} x_j^{\alpha_j} dx = \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-x_j^2} x_j^{\alpha_j} dx_j = 0.$$

Therefore

$$\int_{S^{n-1}} f d\sigma = 0.$$

If every α_i is even, then by symmetry $\int_{-\infty}^{\infty} e^{-x_i^2} x_i^{\alpha_i} dx_i = 2\int_0^{\infty} e^{-x_i^2} x_i^{\alpha_i} dx_i$. The left-hand side is then

$$\int_{\mathbb{R}^n} e^{-|x|^2} \prod_{j=1}^{n} x_j^{\alpha_j} dx = 2^n \prod_{j=1}^{n} \int_0^{\infty} e^{-x_j^2} x_j^{\alpha_j} dx_j = \prod_{j=1}^{n} \int_0^{\infty} e^{-s} s^{\frac{\alpha_j}{2} - \frac{1}{2}} ds = \prod_{j=1}^{n} \Gamma(\frac{\alpha_j + 1}{2}).$$

By combining the identities of the left-hand side and right-hand side, we obtain

$$\int_{S^{n-1}} f d\sigma = \frac{2\Gamma(\beta_1) \cdots \Gamma(\beta_n)}{\Gamma(\beta_1 + \cdots + \beta_n)}$$

where $\beta_j = \frac{\alpha_j + 1}{2}$.