**Folland: Real Analysis, Chapter 3**
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**Problem 3.3**
*Let $\nu$ be a signed measure on $(X, \mathcal{M})$.*
**a.** $L^1(\nu) = L^1(|\nu|)$.
**b.** *If $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$.*
**c.** *If $E \in \mathcal{M}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$*

**Solution:**
**(a)**
Let $f \in L^1(\nu)$. By definition, $f \in L^1(\nu^+) \cap L^1(\nu^-)$. Therefore $\int |f| d\nu^+ < \infty$ and $\int |f| d\nu^- < \infty$. Hence $\int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^- < \infty$ and so $f \in L^1(|\nu|)$.

Consersely, suppose $f \in L^1(|\nu|)$. Then $\int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^- < \infty$. Therefore $\int |f| d\nu^+ < \infty$ and $\int |f| d\nu^- < \infty$. Hence $f \in L^1(\nu^+) \cap L^1(\nu^-)$ and so $f \in L^1(\nu)$.

**(b)**
Let $f \in L^1(\nu)$. Then

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f d\nu^+ - \int f d\nu^- \right| \\ &\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \\ &= \int |f| d|\nu| \end{aligned}$$

**(c)**
Let $E \in \mathcal{M}$. Then if $f$ is a measurable function such that $|f| \leq 1$, we have

$$\left| \int_E f d\nu \right| \leq \int_E |f| d|\nu| \leq \int_E d|\nu| = |\nu|(E).$$

It follows that $\sup\{|\int_E f d\nu| : |f| \leq 1\} \leq |\nu|(E)$.

On the other hand, let $X = P \cup N$ be a Hahn decomposition for $\nu$. Since $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$, we can write

$$\begin{aligned}|\nu|(E) &= \nu^+(E) + \nu^-(E) \\ &= \nu(E \cap P) - \nu(E \cap N) \\ &= \int_E \chi_P d\nu - \int_E \chi_N d\nu \\ &= \int_E (\chi_P - \chi_N) d\nu \\ &= |\int_E (\chi_P - \chi_N) d\nu|\end{aligned}$$

Therefore $|\nu|(E) = |\int_E g d\nu|$ where $g = \chi_P - \chi_N$. Since $|g| \leq 1$, we must have

$$|\nu|(E) \leq \sup\{|\int_E f d\nu| : |f| \leq 1\}.$$

**Problem 3.7**

*Suppose that* $\nu$ *is a signed measure on* $(X, \mathcal{M})$ *and* $E \in \mathcal{M}$.

**(a)** $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M},\ F \subset E\}$ *and* $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M},\ F \subset E\}$

**(b)** $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ are disjoint, and } \cup_1^n E_j = E\}$

**Solution:**

Let $X = P \cup N$ be a Hahn decomposition for $\nu$. Then $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$.

**(a)**

We prove the first statement. Let $F \in \mathcal{M}$, $F \subseteq E$. Then

$$\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) \leq \nu^+(E).$$

It follows that $\sup\{\nu(F) : F \in \mathcal{M},\ F \subseteq E\} \leq \nu^+(E)$.

On the other hand, $\nu^+(E) = \nu(E \cap P)$, and $E \cap P \in \mathcal{M}$, $E \cap P \subseteq E$. So $\nu^+(E) \leq \sup\{\nu(F) : F \in \mathcal{M},\ F \subseteq E\}$. Therefore

$$\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M},\ F \subseteq E\}.$$

We now prove the second statement. Let $F \in \mathcal{M}$, $F \subseteq E$. Then

$$\nu(F) = \nu^+(F) - \nu^-(F) \geq -\nu^-(F) \geq -\nu^-(E).$$

Since $\nu^-(E) \geq -\nu(F)$ for all $F \in \mathcal{M}, F \subseteq E$, then

$$\nu^-(E) \geq \sup\{-\nu(F) : F \in \mathcal{M},\ F \subseteq E\} = -\inf\{\nu(F) : F \in \mathcal{M},\ F \subseteq E\}.$$

On the other hand, $\nu^-(E) = -\nu(E \cap N)$, and $E \cap N \in \mathcal{M}$, $E \cap N \subseteq E$. So

$$\nu^-(E) \le \sup\{-\nu(F) : F \in \mathcal{M},\ F \subseteq E\} = -\inf\{\nu(F) : F \in \mathcal{M},\ F \subseteq E\}.$$

Therefore

$$\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M},\ F \subseteq E\}.$$

**(b)**

Let $n \in \mathbb{N}$, $E_1, \ldots, E_n$ disjoint measurable sets and $\cup_1^n E_j = E$. Then

$$\begin{aligned}
\sum_{j=1}^n |\nu(E_j)| &= \sum_{j=1}^n |\nu^+(E_j) - \nu^-(E_j)| \\
&\le \sum_{j=1}^n |\nu^+(E_j) + \nu^-(E_j)| \\
&= \sum_{j=1}^n |\nu|(E_j) \\
&= |\nu|(E)
\end{aligned}$$

where the last step follows from the fact that $|\nu|$ is a measure. Therefore

$$\sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ are disjoint, and } \cup_1^n E_j = E\} \le |\nu|(E).$$

On the other hand, we have

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = |\nu(E \cap P)| + |\nu(E \cap N)|.$$

Since $E \cap P$ and $E \cap N$ are disjoint and their union is $E$, then

$$|\nu|(E) \le \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ are disjoint, and } \cup_1^n E_j = E\}.$$

**Problem 3.17**

*Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space, $\mathcal{N}$ a sub-$\sigma$-algebra of $\mathcal{M}$, and $\nu = \mu|\mathcal{N}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ (thus $g$ is $\mathcal{N}$-measurable) such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{N}$; if $g'$ is another such function then $g = g'$ $\nu$-a.e. (In probability theory, $g$ is called the conditional expectation of $f$ on $\mathcal{N}$)*

**Solution:**

Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space, $\mathcal{N}$ a sub-$\sigma$-algebra of $\mathcal{M}$, $\nu = \mu|\mathcal{N}$, and $f \in L^1(\mu)$.

We define $\lambda(E) := \int_E f d\mu$ to be a signed measure on $(X, \mathcal{N})$. The fact that $\lambda$ is a signed measure is explained in the first paragraph on page 86, and follows from the fact that at least one of $f^+ d\mu$ and $f^- d\mu$ are finite (indeed, both are finite since $f \in L^1(\mu)$).

Let $A \in \mathcal{N}$. If $\nu(A) = 0$, then $\mu(A) = \nu(A) = 0$, hence $\lambda(A) = 0$. It follows that $\lambda << \nu$.

By the Radon-Nikodym theorem, there exists an extended $\nu$-measurable function $g$ such that $\lambda(E) = \int_E g d\nu$ and any two such functions are equal $\nu$-a.e.. It only remains to show that $g \in L^1(\nu)$. This is clear since

$$\left| \int g d\nu \right| = \left| \int f d\mu \right| \leq \int |f| d\mu < \infty,$$

hence

$$\left| \int g d\nu \right| = \left| \int g^+ d\nu - \int g^- d\nu \right| < \infty.$$

Since $g$ is an extended $\nu$-measurable function, one of $\int g^+ d\nu$, $\int g^- d\nu$ is finite. Since their difference is finite, they must both be finite. Therefore

$$\int |g| d\nu = \int g^+ d\nu + \int g^- d\nu < \infty.$$

**Problem 3.21**

*Let $\nu$ be a complex measure on $(X, \mathcal{M})$. If $E \in \mathcal{M}$, define*

$$\mu_1(E) = \sup\{\sum_1^n |\nu(E_j)| : \ n \in \mathbb{N}, \ E_1, \ldots, E_n \text{ disjoint}, \ E = \bigcup_1^n E_j\}$$

$$\mu_2(E) = \sup\{\sum_1^\infty |\nu(E_j)| : \ n \in \mathbb{N}, \ E_1, \ldots, E_n \text{ disjoint}, \ E = \bigcup_1^\infty E_j\}$$

$$\mu_3(E) = \sup\{\left| \int_E f d\nu \right| : |f| \leq 1\}.$$

*Then $\mu_1 = \mu_2 = \mu_3 = |\nu|$.*

**Solution:** First, we show that $\mu_1 \leq \mu_2$. Let $A_1, \ldots, A_n$ be disjoint sets such that $E = \bigcup_1^n A_j$. Define an infinite collection $\{F_i\}_{i=1}^\infty$ in the following way: for any $i \in \mathbb{N}$, let $F_i = A_i$ if $1 \leq i \leq n$, $F_i = \emptyset$ if $i > n$. Then $F_1, F_2, \ldots$ are disjoint and $E = \bigcup_1^\infty F_j$, hence

$$\sum_1^n |\nu(A_j)| = \sum_1^\infty |\nu(F_j)| \leq \sup\{\sum_1^\infty |\nu(E_j)| : \ n \in \mathbb{N}, \ E_1, \ldots, E_n \text{ disjoint}, \ E = \bigcup_1^\infty E_j\}.$$

It follows that $\mu_1 \leq \mu_2$.

Next, we show that $\mu_2 \leq \mu_3$. Let $A_1, A_2, \ldots$ be disjoint sets such that $E = \cup A_j$. Then

$$
\begin{aligned}
\sum_1^\infty |\nu(A_j)| &\leq \sum_1^\infty |\nu|(A_j) && \text{Proposition 3.13a} \\
&= |\nu|(E) \\
&= \int_E d|\nu| \\
&= \int_E \Big|\frac{d\nu}{d|\nu|}\Big| d|\nu| && \text{Proposition 3.13b} \\
&= \int_E \overline{\frac{d\nu}{d|\nu|}} \frac{d\nu}{d|\nu|} d|\nu| \\
&= \int_E \overline{\frac{d\nu}{d|\nu|}} d\nu && \text{Proposition 3.9a} \\
&\leq \sup\{|\int_E f d\nu| : |f| \leq 1\}
\end{aligned}
$$

Hence $\mu_2 \leq \mu_3$.

Next, we show $\mu_3 \leq \mu_1$. Let $f$ be a measurable function such that $|f| \leq 1$. By Theorem 2.10, there is a sequence $\{\phi_n\}$ of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq 1$, $\phi_n \to f$ pointwise. Since $\nu$ is a complex measure, we know $\nu_r^+, \nu_r^-, \nu_i^+, \nu_i^-$ are finite positive measures, so $\int 1 d\nu_r^+ < \infty$, $\int 1 d\nu_r^- < \infty$, $\int 1 d\nu_i^+ < \infty$, and $\int 1 d\nu_i^- < \infty$. By the Dominated Convergence Theorem, for all $\epsilon > 0$ there exists a simple function $\phi_N = \sum_1^n a_i \chi_{A_i}$ where $|a_i| \leq 1$ and $A_i$ disjoint such that

$$
\begin{aligned}
|\int_E f d\nu| &= |\int_E f d\nu_r^+ - \int_E f d\nu_r^- + i\int_E f d\nu_i^+ - i\int_E f d\nu_i^-| \\
&\leq |\int_E \phi_N d\nu_r^+ + \epsilon - \int_E \phi_N d\nu_r^- + \epsilon + i\int_E \phi_N d\nu_i^+ + i\epsilon + -i\int_E \phi_N d\nu_i^- + \epsilon i| \\
&\leq 4\epsilon + |\int_E \phi_N d\nu| \\
&= 4\epsilon + |\int_E (\sum_1^n a_i \chi_{A_i}) d\nu| \\
&= 4\epsilon + |\sum_1^n a_i \int_E \chi_{A_i} d\nu| \\
&= 4\epsilon + |\sum_1^n a_i \nu(A_i \cap E)| \\
&\leq 4\epsilon + \sum_1^n |\nu(A_i \cap E)| \leq 4\epsilon + |\nu(\bigcap_1^n A_i^c \cap E)| + \sum_1^n |\nu(A_i \cap E)|.
\end{aligned}
$$

Since $A_1 \cap E, \ldots, A_n \cap E, (\bigcap A_i^c \cap E)$ are disjoint and their union is equal to $E$, we have

$$
|\int_E f d\nu| \leq \sup\{\sum_1^n |\nu(E_j)| : \ n \in \mathbb{N}, \ E_1, \ldots, E_n \text{ disjoint}, \ E = \bigcup_1^n E_j\}.
$$

This proves that $\mu_1 = \mu_2 = \mu_3$. To complete the proof we will show that $\mu_3 = |\nu|$.
For any measurable function $f$ such that $|f| \leq 1$, we have

$$|\int_E f d\nu| \leq \int_E |f| d|\nu| \leq |\nu|(E).$$

Hence $\mu_3 \leq |\nu|$. On the other hand, if we let $f = d\nu / d|\nu|$, we can redefine $f$ on a set of $|\nu|$ measure zero such that $|f| = 1$. Then

$$\Big| \int_E \bar{f} d\nu \Big| = \Big| \int_E \bar{f} \frac{d\nu}{d|\nu|} d|\nu| \Big| = \Big| \int_E d|\nu| \Big| = |\nu|(E).$$

Therefore

$$|\nu|(E) \leq \sup\{| \int_E f d\nu | : |f| \leq 1\}.$$

Hence $|\nu| = \mu_3$.

**Problem 3.22**
*If $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exist $C, R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for $|x| > R$. Hence $m(\{x : Hf(x) > \alpha\}) \geq C'/\alpha$ when $\alpha$ is small, so the estimate in the maximal theorem is essentially sharp.*

**Solution:**
Since $f \neq 0$, there exists a $R > 1$ such that

$$\int_{B(R,0)} |f| > c_1 > 0.$$

If $x > |R|$, then

$$Hf(x) \geq \frac{1}{m(B(2|x|, x))} \int_{B(2|x|,x)} |f(y)| dy \geq \frac{1}{m(B(2|x|, x))} \int_{B(R,0)} |f(y)| dy > \frac{c_1 |x|^{-n}}{2^n m(B^n)} = C|x|^{-n},$$

where $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ and $C = \frac{c_1}{2^n m(B^n)}$.
If $\alpha$ is small enough such that $C/\alpha > R^n$, then for $R < |x| < (C/\alpha)^{1/n}$ we have $Hf(x) > \alpha$. Hence

$$m(\{x : Hf(x) > \alpha\}) \geq m(\{x : R < |x| < (C/\alpha)^{1/n}\}) = m(B^n)(\frac{C}{\alpha} - R^n) = \frac{C'}{\alpha},$$

where $C' = Cm(B^n)(1 - \frac{R^n \alpha}{C})$.

**Problem 3.25**
*If $E$ is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of $E$ at $x$ is defined as*

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(r, x))}{m(B(r, x))},$$

*whenever the limit exists.*

**a.** *Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.*

**b.** *Find examples of $E$ and $x$ such that $D_E(x)$ is a given number $\alpha \in (0,1)$, or such that $D_E(x)$ does not exist.*

**Solution:**

**(a)** By Theorem 3.18, $\lim_{r\to 0} A_r\chi_E(x) = \chi_E(x)$ for a.e. $x \in \mathbb{R}^n$. By definition of $A_r$, this implies

$$\lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} \chi_E(y)dy = \lim_{r\to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))} = \chi_E(x)$$

for a.e. $x \in \mathbb{R}^n$. Therefore $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

**(b)** We will find examples in $\mathbb{R}^2$. Let $\alpha \in (0,1)$. Then define

$$E_\alpha = \{(r,\theta) \in \mathbb{R}^2 : 0 < r < 1,\ 0 < \theta < 2\pi\alpha\}.$$

Then when $0 < r < 1$, we have $m(E_\alpha \cap B(r,0)) = \int_0^{2\pi\alpha}\int_0^r s ds d\theta = \pi\alpha r^2$. Therefore

$$\frac{m(E \cap B(r,0))}{m(B(r,0))} = \frac{\pi\alpha r^2}{\pi r^2} = \alpha.$$

So for $E = E_\alpha$ and $x = (0,0)$, $D_E(x) = \alpha$.

Next, we construct an $E$ such that $D_E(x)$ does not exist. Define

$$E_n = [\frac{1}{2^{n+1}}, \frac{1}{2^n}] \times [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$$

and let $E = \bigcup_0^\infty E_n$. Define $r_n = 2^{-n}\sqrt{2}$, and notice $r_n \to 0$ as $n \to \infty$. Since $r_n$ is at the top-right corner of a square, we can sum the area of the squares and compute

$$m(E \cap B(0,r_n)) = \sum_{k=n}^{\infty} \frac{1}{4^k} = \frac{4^{1-n}}{3}.$$

Therefore

$$\frac{m(E \cap B(0,r_n))}{m(B(0,r_n))} = \frac{4^{1-n}2^{2n}}{3(2\pi)} = \frac{2}{3\pi}.$$

Hence

$$\lim_{n\to\infty} \frac{m(E \cap B(0,r_n))}{m(B(0,r_n))} = \frac{2}{3\pi}.$$

However, if we let $\tilde{r}_n = 3 \cdot 2^{-n-1}\sqrt{2}$, then we also have $\tilde{r}_n \to 0$ as $n \to \infty$. Here, each $\tilde{r}_n$ is in the center of a square, so we can get an upper bound on $m(E \cap B(0,\tilde{r}_n))$ by summing the area of all previous squares and adding half of the area of the square where $\tilde{r}_n$ is centered.

$$m(E \cap B(0, \tilde{r}_n)) \leq \frac{1}{4^n}\frac{1}{2} + \sum_{k=n+1}^{\infty} \frac{1}{4^k} = \frac{4^{-n}}{2} + \frac{4^{-n}}{3} = 4^{-n}\frac{5}{6}.$$

This gives an upper bound on the ratio

$$\frac{m(E \cap B(0, \tilde{r}_n))}{m(B(0, \tilde{r}_n))} \leq 4^{-n}\frac{5}{6}\frac{2^{2(n+1)}}{\pi(3\sqrt{2})^2} = \frac{5}{27\pi}.$$

Therefore,

$$\lim_{n\to\infty} \frac{m(E \cap B(0, \tilde{r}_n))}{m(B(0, \tilde{r}_n))} < \lim_{n\to\infty} \frac{m(E \cap B(0, r_n))}{m(B(0, r_n))}.$$

These limits would be equal if $D_E(0)$ existed. Hence the limit as $r \to 0$ of ratio between $m(E \cap B(0, r))$ and $m(B(0, r)$ does not exist.

**Problem 3.31**

*Let $F(x) = x^2 \sin(x^{-1})$ and $G(x) = x^2 \sin(x^{-2})$ for $x \neq 0$, and $F(0) = G(0) = 0$.*

**a.** *$F$ and $G$ are differentiable everywhere (including $x = 0$).*

**b.** *$F \in BV([-1, 1])$, but $G \notin BV([-1, 1])$.*

**Solution:**

**(a)** It is clear that $F$ and $G$ are differentiable at $x \neq 0$ since they are the product and composition of differentiable functions. At $x = 0$ we can use the definition of the derivative:

$$|F'(0)| = \lim_{h\to 0} \left|\frac{F(0+h) - F(0)}{h - 0}\right| \leq \lim_{h\to 0} \left|\frac{h^2 \sin(1/h)}{h}\right| \leq \lim_{h\to 0} |h| = 0.$$

Hence $F'(0) = 0$, and by the same argument $G'(0) = 0$.

**(b)**

For $x \neq 0$, we have $F'(x) = 2x \sin(1/x) - \cos(1/x)$. Therefore, on $[-1, 1]$ we have $|F'(x)| \leq 2(1)(1) + 1 = 3$. For any subdivision $n \in \mathbb{N}, -1 = x_0 < \cdots < x_n = 1$, we can apply the Mean Value Theorem to conclude

$$\sum_{1}^{n} |F(x_j) - F(x_{j-1})| \leq \sum_{1}^{n} 3|x_j - x_{j-1}| = 6.$$

Therefore $F \in BV([-1, 1])$.

Next we show that $G \notin BV([-1, 1])$. Define $x_k = (\pi k + \pi/2)^{-1/2}$ and let $k \in \mathbb{N}, k > 2$ and consider the following partition $P_k$ of $[-1, 1] : -1, -x_1, -x_2, \ldots, -x_k, 0, x_k, \ldots, x_2, x_1$. Then

$$\begin{aligned}\sum_{P_k}|G(x_i)-G(x_{i-1})| &\geq \sum_{n=2}^{k}\Big|\frac{1}{\pi n+\pi/2}\sin(\pi n+\pi/2)-\frac{1}{\pi(n-1)+\pi/2}\sin(\pi(n-1)+\pi/2)\Big| \\ &= \sum_{n=2}^{k}\Big|\frac{1}{\pi n+\pi/2}+\frac{1}{\pi(n-1)+\pi/2}\Big| \\ &\geq \frac{1}{\pi}\sum_{n=2}^{k}\frac{1}{n+1/2}\end{aligned}$$

Since the harmonic series diverges, if we refine $P_k$ by increasing $k$, this sum can be made as large as we like. Hence $\sup\{\sum_1^n |G(x_j)-G(x_{j-1})| : n \in \mathbb{N}, -1 = x_0 < \cdots < x_n = 1\} = \infty$ and $G \notin BV([-1,1])$.

**Problem 3.37**

*Suppose $F : \mathbb{R} \to \mathbb{C}$. There is a constant $M$ such that $|F(x)-F(y)| \leq M|x-y|$ for all $x, y \in \mathbb{R}$ (that is, $F$ is Lipschitz continuous) iff $F$ is absolutely continuous and $|F'| \leq M$ a.e.*

**Solution:**

Suppose $F$ is Lipschitz continuous. Let $\epsilon > 0$, and $\delta = \epsilon/M$. Then for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$ such that $\sum_1^N (b_j - a_j) < \delta$, we have

$$\sum_1^N |F(b_j)-F(a_j)| \leq \sum_1^N M|b_j - a_j| < \delta M = \epsilon.$$

Hence $F$ is absolutely continuous. Also,

$$|F'(x)| = \lim_{y\to x}\frac{|F(x)-F(y)|}{|x-y|} \leq M.$$

Conversely, suppose $F$ is absolutely continuous and $|F'| \leq M$ a.e. If $x, y \in \mathbb{R}$ and $x > y$, by the Fundamental Theorem of Calculus for Lebesgue Integrals, we have

$$|F(x)-F(y)| = \Big|\int_y^x F'(t)dt\Big| \leq \int_y^x |F'(t)|dt \leq M(x-y).$$

It follows that $|F(x)-F(y)| \leq M|x-y|$ for all $x, y \in \mathbb{R}$.