Folland: Real Analysis, Chapter 4
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Problem 4.19

If $\{X_\alpha\}$ is a family of topological spaces, $X = \prod_\alpha X_\alpha$ (with the product topology) is uniquely determined up to homeomorphism by the following property: There exist continuous maps $\pi_\alpha : X \to X_\alpha$ such that if Y is any topological space and $f_\alpha \in C(Y, X_\alpha)$ for each α, there is a unique $F \in C(Y, X)$ such that $f_\alpha = \pi_\alpha \circ F$.

Solution:

First, we show that if $X = \prod_{\alpha \in A} X_\alpha$ with the product topology, then the coordinate maps $\pi_\alpha : X \to X_\alpha$ are such that if Y is any topological space and $f_\alpha \in C(Y, X_\alpha)$ for each α, there is a unique $F \in C(Y, X)$ such that $f_\alpha = \pi_\alpha \circ F$.

Indeed, define $F : Y \to X$ in the following way: for any $y \in Y$, let $F(y) \in \prod_{\alpha \in A} X_\alpha$ be the map $F(y) : A \to \bigcup_{\alpha \in A} X_\alpha$ such that $(F(y))_\alpha = f_\alpha(y)$ for each $\alpha \in A$. It then follows from the definition of the coordinate maps that for any $y \in Y$, $\pi_\alpha \circ F(y) = (F(y))_\alpha = f_\alpha(y)$. Since the f_α are continuous, $\pi_\alpha \circ F$ is continuous for each α, hence F is continuous by Proposition 4.11.

For uniqueness, let $F' \in C(Y, X)$ be such that $f_\alpha = \pi_\alpha \circ F'$. Suppose there exists $y \in Y$ such that $F'(y) \neq F(y)$. Then there is an $\alpha \in A$ such that $(F'(y))_\alpha \neq (F(y))_\alpha$. But then

$$f_\alpha(y) = \pi_\alpha \circ F'(y) = (F'(y))_\alpha \neq (F(y))_\alpha = \pi_\alpha \circ F(y) = f_\alpha(y).$$

This contradiction shows that $F' = F$.

Next, we show that up to homeomorphism, the product topology is the only space with this property. Let X' be a topological space such that there exists continuous maps $\pi'_\alpha : X' \to X_\alpha$ such that if Y is any topological space and $f_\alpha \in C(Y, X_\alpha)$ for each α, there is a unique $F \in C(Y, X')$ such that $f_\alpha = \pi'_\alpha \circ F$.

Let $G : X \to X'$ be the continuous function such that $\pi_\alpha = \pi'_\alpha \circ G$ and $G' : X' \to X$ be the continuous function such that $\pi'_\alpha = \pi_\alpha \circ G'$. To complete the proof, we will show that $G \circ G' = G' \circ G = id$. We have

$$\pi_\alpha \circ (G' \circ G) = (\pi_\alpha \circ G') \circ G = \pi'_\alpha \circ G = \pi_\alpha,$$

hence $\pi_\alpha = \pi_\alpha \circ id = \pi_\alpha \circ (G' \circ G)$. Since $\pi_\alpha \in C(X, X_\alpha)$, by uniqueness we must have $id = G' \circ G$. Similarly,

$$\pi'_\alpha \circ (G \circ G') = (\pi'_\alpha \circ G) \circ G' = \pi_\alpha \circ G' = \pi'_\alpha \circ id,$$

and by uniquenes we must have $id = G \circ G'$.

Problem 4.22
Let X be a topological space, (Y,ρ) a complete metric space, and $\{f_n\}$ a sequence in Y^X such that $\sup_{x\in X}\rho(f_n(x),f_m(x))\to 0$ as $m,n\to\infty$. There is a unique $f\in Y^X$ such that $\sup_{x\in X}\rho(f_n(x),f(x))\to 0$ as $n\to\infty$. If each f_n is continuous, so is f.

Solution:
For each $x\in X$, we have $\rho(f_n(x),f_m(x))\le\sup_{z\in X}\rho(f_n(z),f_m(z))\to 0$ as $m,n\to\infty$. Since (Y,ρ) is complete, $\{f_n(x)\}$ is a Cauchy sequence, hence converges to a point in Y which we may call $f(x)$. Therefore $\{f_n\}$ converges pointwise to a function $f\in Y^X$. For any $\epsilon>0$, there exists a positive integer N such that when $m,n>N$ we have $\rho(f_n(x),f_m(x))\le\epsilon$ for all $x\in X$. If we fix n and let $m\to\infty$, we obtain

$$\rho(f_n(x),f(x))\le\epsilon.$$

Since this holds for all $x\in X$, we have $\sup_{x\in X}\rho(f_n(x),f(x))\to 0$ as $n\to\infty$.

Suppose there exists another function $g\in Y^X$ such that $\sup_{x\in X}\rho(f_n(x),g(x))\to 0$ as $n\to\infty$. If for some $x\in X$, $f(x)\ne g(x)$, then

$$\lim_{n\to\infty}f_n(x)=f(x)\ne g(x)=\lim_{n\to\infty}f_n(x),$$

which contradicts uniqueness of the limit of a sequence in a metric space.

Suppose each f_n is continuous. Let $\epsilon>0$ and let $x\in X$. Choose a positive integer N such that $\rho(f_N(z),f(z))<\epsilon/3$ for all $z\in X$. Choose $\delta>0$ such that $\rho(f_N(x),f_N(y))<\epsilon/3$ for all $y\in X$ such that $\rho(x,y)<\delta$. Then we have

$$\rho(f(x),f(y))\le\rho(f(x),f_N(x))+\rho(f_N(x),f_N(y))+\rho(f_N(y),f(y))\le\epsilon/3+\epsilon/3+\epsilon/3=\epsilon,$$

for all $y\in Y$ such that $\rho(x,y)<\delta$. This proves continuity of f.

Problem 4.28
Let X be a topological space equipped with an equivalence relation, $\tilde X$ the set of equivalence classes, $\pi: X\to\tilde X$ the map taking each $x\in X$ to its equivalence class, and $\mathcal{T}=\{U\subset\tilde X:\pi^{-1}(U)\text{ is open in }X\}$.

a. *$\mathcal{T}$ is a topology on $\tilde X$. (It is called the quotient topology.)*

b. *If Y is a topological space, $f:\tilde X\to Y$ is continuous iff $f\circ\pi$ is continuous.*

c. *$\tilde X$ is T_1 iff every equivalence class is closed.*

Solution:
a. Since $\pi^{-1}(\emptyset)=\emptyset$, then $\emptyset\in\mathcal{T}$, and since π is a surjection, $\pi^{-1}(\tilde X)=X$ hence $\tilde X\in\mathcal{T}$.
Next, suppose $\{U_\alpha\}_{\alpha\in J}\in\mathcal{T}$. Then

$$\pi^{-1}(\bigcup_{\alpha\in J}U_\alpha)=\bigcup_{\alpha\in J}\pi^{-1}(U_\alpha),$$

hence $\bigcup_{\alpha\in J}U_\alpha\in\mathcal{T}$. Similarly, if $U_i\in\mathcal{T}$ for $i\in\{1,\dots,n\}$, then

$$\pi^{-1}(\bigcap_{i=1}^n U_i)=\bigcap_{i=1}^n\pi^{-1}(U_i).$$

b. The quotient topology is defined such that π is continuous. Suppose $f : \tilde{X} \to Y$ is continuous. Then $f \circ \pi$ is the composition of continuous functions, hence is continuous. Conversely, suppose $f \circ \pi$ is continuous. Then for all open sets $V \subset Y$, $\pi^{-1} \circ f^{-1}(V)$ is open in X, so $f^{-1}(V)$ is in $\mathcal{T}$. Therefore f is continuous.

c. Suppose $\tilde{X}$ is T_1. Then $\{x\}$ is closed for every $x \in \tilde{X}$ (Proposition 4.7). By continuity, $\pi^{-1}(\{x\})$ is closed. Therefore equivalence classes are closed. Conversely, suppose every equivalence class is closed. Then $\pi^{-1}(\{x\})$ is closed for every $x \in \tilde{X}$. Then $(\pi^{-1}(\{x\}))^c = \pi^{-1}(\{x\}^c)$ is open, hence $\{x\}^c \in \mathcal{T}$, so $\{x\}$ is closed in $\tilde{X}$. Therefore $\tilde{X}$ is T_1 (Proposition 4.7).

Problem 4.32

A topological space X is Hausdorff iff every net in X converges to at most one point.

Solution:

Suppose X is Hausdorff and there is a net $\langle x_\alpha \rangle_{\alpha \in A}$ that converges to two points x, y. There exists disjoint open sets U, V such that $x \in U, y \in V$. But then there exists $\alpha_1 \in A$ such that $x_\alpha \in U$ for all $\alpha \gtrsim \alpha_1$ and $\alpha_2 \in A$ such that $x_\alpha \in V$ for all $\alpha \gtrsim \alpha_2$. There exists a $\gamma \in A$ such that $\gamma \gtrsim \alpha_1$ and $\gamma \gtrsim \alpha_2$, so $\alpha_\gamma \in U \cap V$. This contradiction shows that every net in X converges to at most one point.

Conversely, suppose X is not Hausdorff. Then there exists two points $x, y \in X$ such that every pair of open sets U, V such that $x \in U, y \in V$ has a non-empty intersection. Consider the directed set $\mathcal{N}_x \times \mathcal{N}_y$, where $\mathcal{N}_x, \mathcal{N}_y$ are the families of open neighbourhoods of x, y and $(U_1, V_1) \lesssim (U_2, V_2)$ iff $U_1 \supset U_2$ and $V_1 \supset V_2$. Define the net $\langle x_\alpha \rangle_{\alpha \in \mathcal{N}_x \times \mathcal{N}_y}$ by mapping $(U, V) \in \mathcal{N}_x \times \mathcal{N}_y$ to a point $z \in U \cap V$.

For any open set $\tilde{U}$ containing x, let $\tilde{V}$ be an arbitrary open set containing y. It follows that for every $(U, V) \gtrsim (\tilde{U}, \tilde{V})$ we have $x_{(U,V)} \in \tilde{U}$. Hence $\langle x_\alpha \rangle_{\alpha \in \mathcal{N}_x \times \mathcal{N}_y}$ converges to x. Similarly, for any open set $\tilde{V}$ containing y, let $\tilde{U}$ be an arbitrary open set containing x. It follows that for every $(U, V) \gtrsim (\tilde{U}, \tilde{V})$ we have $x_{(U,V)} \in \tilde{V}$. Hence $\langle x_\alpha \rangle_{\alpha \in \mathcal{N}_x \times \mathcal{N}_y}$ converges to y. Therefore there exists a net in X which converges to more than one point.

Problem 4.34

If X has the weak topology generated by a family $\mathcal{F}$ of functions, then $\langle x_\alpha \rangle$ converges to $x \in X$ iff $\langle f(x_\alpha) \rangle$ converges to $f(x)$ for all $f \in \mathcal{F}$.

Solution:

Suppose $\langle x_\alpha \rangle$ converges to $x \in X$. Since every $f \in \mathcal{F}$ is continuous, $\langle f(x_\alpha) \rangle$ converges to $f(x)$ for all $f \in \mathcal{F}$ by Proposition 4.19.

Conversely, suppose $\langle f(x_\alpha) \rangle$ converges to $f(x)$ for all $f \in \mathcal{F}$. Let U be an open set such that $x \in U$. By the definition weak topology generated by $\mathcal{F}$, there exists an open set $U' \subset U$ such that $x \in U'$ and

$$U' = \bigcap_{i=1}^{n} f_i^{-1}(V_i)$$

where $f_i \in \mathcal{F}$ and V_i are open sets in the target space of f_i. Then for each integer i between 1 and n, there exists an α_i such that $f_i(x_\alpha) \in V_i$ for all $\alpha \gtrsim \alpha_i$. Take $\gamma \gtrsim \alpha_i$ for all integers i between 1 and n. Then when $\alpha \gtrsim \gamma$, we have $x_\alpha \in f_i^{-1}(V_i)$ for all integers i between 1 and n. Hence $x_\alpha \in U' \subset U$ for all $\alpha \gtrsim \gamma$.

Problem 4.52

The one-point compactification of $\mathbb{R}^n$ *is homeomorphic to the n-sphere* $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

Solution:

Denote $N = (0, \ldots, 0, 1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Then the stereographic projection $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ defined by

$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}$$

is a homeomorphism. It is easy to see that the mapping is continuous. Its inverse is given by the continous map

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

Indeed, $\sigma \circ \sigma^{-1} = id$:

$$\begin{aligned}
\sigma \circ \sigma^{-1}(u^1, \ldots, u^n) &= \sigma\left(\frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}\right) \\
&= \frac{(2u^1, \ldots, 2u^n)}{(|u|^2 + 1)(1 - (|u|^2 - 1)(|u|^2 + 1)^{-1})} \\
&= \frac{|u|^2 + 1}{(|u|^2 + 1)(|u|^2 + 1 - (|u|^2 - 1)}(2u^1, \ldots, 2u^n) \\
&= (u^1, \ldots, u^n)
\end{aligned}$$

Also, $\sigma^{-1} \circ \sigma = id$:

$$\begin{aligned}
\sigma^{-1} \circ \sigma(x^1, \ldots, x^{n+1}) &= \sigma^{-1}\left(\frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}\right) \\
&= \left(1 + \sum_{i=1}^n \frac{(x^i)^2}{(1 - x^{n+1})^2}\right)^{-1} \left(\frac{2x^1}{1 - x^{n+1}}, \ldots, \frac{2x^n}{1 - x^{n+1}}, \sum_{i=1}^n \frac{(x^i)^2}{(1 - x^{n+1})^2} - 1\right) \\
&= \frac{(1 - x^{n+1})^2}{(1 - x^{n+1})^2 + \sum_{i=1}^n (x^i)^2}\left(\frac{2x^1}{1 - x^{n+1}}, \ldots, \frac{2x^n}{1 - x^{n+1}}, \sum_{i=1}^n \frac{(x^i)^2 - (1 - x^{n+1})^2}{(1 - x^{n+1})^2}\right) \\
&= \frac{(1 - x^{n+1})^2}{(1 - x^{n+1})^2 + 1 - (x^{n+1})^2}\left(\frac{2x^1}{1 - x^{n+1}}, \ldots, \frac{2x^n}{1 - x^{n+1}}, \frac{1 - (x^{n+1})^2 - (1 - x^{n+1})^2}{(1 - x^{n+1})^2}\right) \\
&= \frac{(1 - x^{n+1})^2}{2 - 2x^{n+1}}\left(\frac{2x^1}{1 - x^{n+1}}, \ldots, \frac{2x^n}{1 - x^{n+1}}, \frac{2x^{n+1} - 2(x^{n+1})^2}{(1 - x^{n+1})^2}\right) \\
&= (x^1, \ldots, x^{n+1})
\end{aligned}$$

Hence we have shown that $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n \backslash \{N\}$. It follows that their one-point compactifications are homeomorphic. It is clear from the definition that the one-point compactification of $\mathbb{S}^n \backslash \{N\}$ is $\mathbb{S}^n$, hence the one-point compactification of $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n$.

Problem 4.60
The product of countably many sequentially compact spaces is sequentially compact.

Solution:
Let $X = \prod_{i\in\mathbb{N}} X_i$, and let $\{x_j\}_{j=1}^{\infty}$ be a sequence in X. Then there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_j\}$ such that $\{\pi_1(x_{n_j}))\}_{j=1}^{\infty}$ converges in X_1. Denote it by $\{x_j^1\}_{j=1}^{\infty}$. Proceeding inductively, for $k \in \mathbb{N}$ we obtain a subsequence $\{x_j^k\}_{j=1}^{\infty}$ of $\{x_j^{k-1}\}_{j=1}^{\infty}$ such that $\{\pi_k(x_j^k)\}_{j=1}^{\infty}$ converges is X_k. Let $y_k = x_k^k$. Then $\{y_k\}_{k=1}^{\infty}$ is a subsequence of $\{x_j\}_{j=1}^{\infty}$ such that $\{\pi_i(y_k)\}_{k=1}^{\infty}$ converges in X_i for all positive integers i. By Problem 4.34 (proved above), we can conclude that $\{y_k\}_{k=1}^{\infty}$ converges in X.

Problem 4.64
Let (X,ρ) be a metric space. A function $f \in C(X)$ is called Holder continuous of exponent α ($\alpha > 0$) if the quantity

$$N_\alpha(f) = \sup_{x\neq y} \frac{|f(x)-f(y)|}{\rho(x,y)^\alpha}$$

is finite. If X is compact, $\{f \in C(X) : ||f||_u \leq 1 \text{ and } N_\alpha(F) \leq 1\}$ is compact in $C(X)$.

Solution:
Denote $\mathcal{F} = \{f \in C(X) : ||f||_u \leq 1 \text{ and } N_\alpha(F) \leq 1\}$. It is clear from the definition that $\mathcal{F}$ is pointwise bounded. Let $\epsilon > 0, x \in X$, and $\delta = \epsilon^{1/\alpha}$. Then for all $y \in X$ such that $\rho(x,y) < \delta$, we have

$$|f(x) - f(y)| \leq \rho(x,y)^\alpha < \epsilon$$

for all $f \in \mathcal{F}$. Hence $\mathcal{F}$ is equicontinuous.

By Arzela-Ascoli, the closure of $\mathcal{F}$ in $C(X)$ is compact. We show that $\mathcal{F}$ is equal to its closure in $C(X)$ to complete the proof.

Suppose $\{f_n\} \subset \mathcal{F}$ and $||f_n - f||_u \to 0$. We will show that $f \in \mathcal{F}$. For all $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that

$$||f||_u \leq ||f - f_n||_u + ||f_n||_u \leq 1 + \epsilon.$$

Since this holds for all $\epsilon > 0$, we can conclude that $||f||_u \leq 1$.

Next, fix $x, y \in X$. Let $\epsilon > 0$, and choose $n \in \mathbb{N}$ such that $||f_n - f||_u < \epsilon/2$. Then

$$\begin{aligned}
\frac{|f(x)-f(y)|}{\rho(x,y)^\alpha} &\leq \frac{|f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)|}{\rho(x,y)^\alpha} \\
&\leq \frac{\epsilon}{\rho(x,y)^\alpha} + \frac{|f_n(x)-f_n(y)|}{\rho(x,y)^\alpha} \\
&\leq 1 + \frac{\epsilon}{\rho(x,y)^\alpha}
\end{aligned}$$

Since this holds for all $\epsilon > 0$, we can conclude that

$$\frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \leq 1.$$

This is true for all $x, y \in X$, therefore it follows that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \leq 1.$$

Hence $f \in \mathcal{F}$, and therefore $\mathcal{F}$ is closed in $C(X)$.