Folland: Real Analysis, Chapter 5
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Problem 5.7
Let X be a Banach space.

a. *If $T \in L(X,X)$ and $||I-T|| < 1$ where I is the identity operator, then T is invertible; in fact, the series $\sum_0^\infty (I-T)^n$ converges in $L(X,X)$ to T^{-1}.*

b. *If $T \in L(X,X)$ is invertible and $||S-T|| < ||T^{-1}||^{-1}$, then S is invertible. Thus the set of invertible operators in open in $L(X,X)$.*

Solution:
(a) First, we notice that $\sum_0^\infty (I-T)^n$ converges in $L(X,X)$. Since $\gamma = ||I-T|| < 1$,

$$\sum_{n=0}^\infty ||(I-T)^n|| \leq \sum_{n=0}^\infty ||I-T||^n = \frac{1}{1-\gamma} < \infty.$$

Therefore, $\sum_0^\infty (I-T)^n$ converges absolutely. Since X is complete, so is $L(X,X)$, and therefore $\sum_0^\infty (I-T)^n$ converges in $L(X,X)$. Denote $X = \sum_0^\infty (I-T)^n$.

Next, we show that TX and XT are equal to the identity to conclude that T has a two-sided inverse and is a bijection. First, we derive the following:

$$(I-T)X = \sum_{n=0}^\infty (I-T)^{n+1} = \sum_{n=1}^\infty (I-T)^n = \sum_{n=0}^\infty (I-T)^n - I = X - I.$$

It follows that $TX = I$. The same calculation yields

$$X(I-T) = \sum_{n=0}^\infty (I-T)^{n+1} = X - I.$$

Therefore T has a two-sided inverse and is a bijection.

To complete the proof, we must show that $T^{-1} = X$ is bounded. Denote the partial sums as $S_n = \sum_{i=0}^n (I-T)^i$. Then, using continuity of the norm to exchange the limit, we obtain

$$||T^{-1}x|| = ||\lim S_n x|| = \lim ||S_n x|| \leq \lim \sum_{i=0}^n ||I-T||^i ||x|| = \frac{1}{1-\gamma} ||x||.$$

Therefore, $||T^{-1}|| < \infty$.

(b) We have

$$||ST^{-1} - I|| = ||(ST^{-1} - I)TT^{-1}|| \leq ||S-T|| \cdot ||T^{-1}|| < 1.$$

From part (a), we conclude that $ST^{-1} = A \in L(X, X)$ is invertible. Then $S = AT$ is the product of two invertible operators. S is thus a bijection with inverse $T^{-1}A^{-1}$, and is bounded since

$$||S^{-1}|| \leq ||T^{-1}|| \cdot ||A^{-1}||.$$

Problem 5.10

Let $L_k^1([0,1])$ be the space of all $f \in C^{k-1}([0,1])$ such that $f^{(k-1)}$ is absolutely continuous on $[0,1]$ (and hence $f^{(k)}$ exists a.e. and is in $L^1([0,1])$. Then $||f|| = \sum_0^k \int_0^1 |f^{(j)}(x)|dx$ is a norm on $L_k^1([0,1])$ that makes $L_k^1([0,1])$ into a Banach space.

Solution:

It is immediate that $L_k^1([0,1])$ is a normed vector space. The hard part is to show that $L_k^1([0,1])$ is complete.

First, we deal with the case $k = 1$. Let $\sum_1^\infty f_n$ be an absolutely convergent series in $L_1^1([0,1])$. Written explicitely, this means that

$$\sum_{n=1}^{\infty} ||f_n||_1^1 = \sum_{n=1}^{\infty} \left(\int_0^1 |f_n(x)|dx + \int_0^1 |f_n'(x)|dx \right) < \infty.$$

Therefore, $\sum_1^\infty \int |f_n| < \infty$, and by Theorem 2.25, $\sum_1^\infty f_n$ converges a.e. to a function in $L^1([0,1])$. By the fundamental theorem of calculus, for some $a \in [0,1]$ we have

$$\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{\infty} \left| \int_a^x f_n'(t)dt + f_n(a) \right| \leq \sum_{n=1}^{\infty} \int_a^x |f_n'(t)dt| + |f_n(a)| < \infty.$$

By completeness of the norm $|.|$, we see that $\sum_1^\infty f_n(x)$ converges with respect to $|.|$ for each $x \in [0,1]$.

Now, we also have that $\sum_1^\infty \int |f_n'| < \infty$, so again by Theorem 2.25, $\sum_1^\infty f_n'$ converges a.e. to a function $g \in L^1([0,1])$. By invoking Theorem 2.25 again and the fundamental theorem of calculus, we see that for all $x \in [0,1]$,

$$\begin{aligned} \int_0^x g(t)dt &= \int_0^x \left(\sum_{n=1}^{\infty} f_n'(t) \right) dt \\ &= \sum_{n=1}^{\infty} \int_0^x f_n'(t)dt \\ &= \sum_{n=1}^{\infty} f_n(x) - f_n(0) \\ &= f(x) - f(0). \end{aligned}$$

Therefore, f is absolutely continuous on $[0,1]$, and furthermore $g(t) = f'(t)$ a.e. We can now see that $\sum_1^\infty f_n$ is a convergent series in $L_1^1([0,1])$:

$$
\begin{aligned}
||f - \sum_{n=1}^{N} f_n||_1^1 &= \int_0^1 |f - \sum_{n=1}^{N} f_n| + \int_0^1 |f' - \sum_{n=1}^{N} f'_n| \\
&\leq \int_0^1 \sum_{n=N}^{\infty} |f_n| + \int_0^1 \sum_{n=N}^{\infty} |f'_n| \\
&= \sum_{n=N}^{\infty} \Big(\int_0^1 |f_n| + \int_0^1 |f'_n| \Big),
\end{aligned}
$$

which goes to zero as $n \to \infty$. By Theorem 5.1, $L_1^1([0,1])$ is complete.

We proceed by induction on k. The arguments are similar to the ones above, so they will not be repeated in detail. Let $\sum_1^\infty f_n$ be an absolutely convergent series in $L_{k+1}^1([0,1])$. Then $\sum_1^\infty f_n$ is an absolutely convergent series in $L_k^1([0,1])$, and by induction hypothesis, $\sum_1^\infty f_n$ converges to a function f with respect to $L_k^1([0,1])$. Furthermore, $\sum_1^\infty f_n^{(k)}$ converges a.e. to $f^{(k)}$ with respect to $|.|$. By the fundamental theorem of calculus, for some $a \in [0,1]$ we have

$$
\sum_{n=1}^{\infty} |f_n^{(k)}(x)| = \sum_{n=1}^{\infty} \Big| \int_a^x f_n^{(k+1)}(t)dt + f_n^{(k)}(a) \Big| \leq \sum_{n=1}^{\infty} \int_a^x |f_n^{(k+1)}(t)dt| + |f_n^{(k)}(a)| < \infty.
$$

By completeness of the norm $|.|$, we see that $\sum_1^\infty f_n^{(k)}(x)$ converges with respect to $|.|$ for each $x \in [0,1]$. We can now repeat the same argument as done before to show that $f^{(k)}$ is absolutely continuous and $\sum_1^\infty f_n^{(k+1)}$ converges almost everywhere to $f^{(k+1)}$. Then

$$
\begin{aligned}
||f - \sum_{n=1}^{N} f_n||_{k+1}^1 &= ||f - \sum_{n=1}^{N} f_n||_k^1 + \int_0^1 |f^{(k+1)} - \sum_{n=1}^{N} f_n^{(k+1)}| \\
&\leq ||f - \sum_{n=1}^{N} f_n||_k^1 + \sum_{n=N}^{\infty} \int_0^1 |f_n^{(k+1)}|,
\end{aligned}
$$

which goes to zero as $N \to \infty$, showing completeness of $L_{k+1}^1([0,1])$.

Problem 5.11

If $0 < \alpha \leq 1$, let $\Lambda_\alpha([0,1])$ be the space of Holder continuous functions of exponent α on $[0,1]$. That is, $f \in \Lambda_\alpha([0,1])$ iff $||f||_{\Lambda_\alpha} < \infty$, where

$$
||f||_{\Lambda_\alpha} = |f(0)| + \sup_{x,y\in[0,1], x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.
$$

a. *$||\cdot||_{\Lambda_\alpha}$ is a norm that makes $\Lambda_\alpha([0,1])$ into a Banach space.*

b. *Let $\lambda_\alpha([0,1])$ be the set of all $f \in \Lambda_\alpha([0,1])$ such that*

$$
\frac{|f(x) - f(y)|}{|x-y|^\alpha} \to 0 \text{ as } x \to y, \text{ for all } y \in [0,1].
$$

If $\alpha < 1$, $\lambda_\alpha([0,1])$ is an infinite-dimensional closed subspace of $\Lambda_\alpha([0,1])$. If $\alpha = 1$, $\lambda_\alpha([0,1])$ contains only constant functions.

Solution:
a. We first show the triangle inequality. Let $f, g \in \Lambda_\alpha([0,1])$.

$$\begin{aligned} ||f+g||_{\Lambda_\alpha} &= |f(0)+g(0)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)+g(x)-f(y)-g(y)|}{|x-y|^\alpha} \\ &\leq |f(0)| + |g(0)| + \sup_{x,y\in[0,1],x\neq y} \left(\frac{|f(x)-f(y)|}{|x-y|^\alpha} + \frac{|g(x)-g(y)|}{|x-y|^\alpha} \right) \\ &\leq ||f||_{\Lambda_\alpha} + ||g||_{\Lambda_\alpha}. \end{aligned}$$

Scalar multiplication is shown similarly. If $||f||_{\Lambda_\alpha} = 0$, then $|f(0)| = 0$ and $|f(x)|/|x|^\alpha = 0$ for all $x \neq 0$. Hence $f = 0$.

We now show completeness using Theorem 5.1. Let $\{f_n\}$ be a sequence in $\Lambda_\alpha([0,1])$ such that $\sum_1^\infty ||f_n||_{\Lambda_\alpha} < \infty$. Then for any $x \in (0,1]$, we have

$$\begin{aligned} \sum_{n=1}^\infty |f(x)| &\leq \sum_{n=1}^\infty |f(0)| + |f(x) - f(0)| \\ &\leq \sum_{n=1}^\infty |f(0)| + \frac{|f(x)-f(0)|}{|x|^\alpha} \\ &\leq \sum_{n=1}^\infty ||f||_{\Lambda_\alpha} < \infty. \end{aligned}$$

When $x = 0$, we have $\sum_{n=1}^\infty |f(0)| \leq \sum_{n=1}^\infty ||f||_{\Lambda_\alpha} < \infty$.

Therefore, $\sum f_n(x)$ converges absolutely with respect to $|\cdot|$ for all $x \in [0,1]$. Define

$$f(x) := \sum_{n=1}^\infty f_n(x).$$

We show $f \in \Lambda_\alpha([0,1])$. Let $x, y \in [0,1]$, $x \neq y$. Then

$$\begin{aligned} \frac{|f(x)-f(y)|}{|x-y|^\alpha} &= \frac{|\sum_{n=1}^\infty f_n(x) - f_n(y)|}{|x-y|^\alpha} \\ &\leq \sum_{n=1}^\infty \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} \\ &\leq \sum_{n=1}^\infty ||f_n||_{\Lambda_\alpha} < \infty. \end{aligned}$$

This shows that

$$\sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

Since $|f(0)| < \infty$, we see that $||f||_{\Lambda_\alpha} < \infty$. To complete the proof, we show $\sum f_n \to f$ with respect to $||\cdot||_{\Lambda_\alpha}$:

$$||\sum_{n=1}^{N} f_n - f||_{\Lambda_\alpha} = ||\sum_{n=N}^{\infty} f_n||_{\Lambda_\alpha} \leq \sum_{n=N}^{\infty} ||f_n||_{\Lambda_\alpha} \to 0,$$

as $N \to \infty$.

b. Let $\alpha < 1$. First, we show $\lambda_\alpha([0,1])$ is a subspace of $\Lambda_\alpha([0,1])$. If $f, g \in \lambda_\alpha([0,1])$, $c \in \mathbb{R}$, then for all $y \in [0,1]$:

$$\frac{|f(x)+cg(x)-f(y)-cg(y)}{|x-y|^\alpha} \leq \frac{|f(x)-f(y)|}{|x-y|^\alpha} + |c|\frac{|g(x)-g(y)|}{|x-y|^\alpha} \to 0,$$

as $x \to y$ since $\frac{|f(x)-f(y)|}{|x-y|^\alpha} \to 0$ and $\frac{|g(x)-g(y)|}{|x-y|^\alpha} \to 0$ by assumption. Therefore $\lambda_\alpha([0,1])$ is a subspace of $\Lambda_\alpha([0,1])$.

We now show that $\lambda_\alpha([0,1])$ is closed. Suppose $\{f_n\} \in \lambda_\alpha([0,1])$ and $f_n \to f$ with respect to $||\cdot||_{\Lambda_\alpha}$. Then there exists $N \in \mathbb{N}$ such that $||f_N - f||_{\Lambda_\alpha} < \epsilon/2$. Let $y \in [0,1]$. Also, there exists $\delta > 0$ such that when $|x-y| < \delta$, we have

$$\frac{|f_N(x)-f_N(y)|}{|x-y|^\alpha} < \epsilon/2.$$

Hence, when $|x-y| < \delta$, we get

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|^\alpha} &\leq \frac{|f(x)-f_N(x)-(f(y)-f_N(y))|}{|x-y|^\alpha} + \frac{|f_N(x)-f_N(y)|}{|x-y|^\alpha} \\ &\leq ||f_N - f||_{\Lambda_\alpha} + \frac{|f_N(x)-f_N(y)|}{|x-y|^\alpha} \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Therefore, $\frac{|f(x)-f(y)|}{|x-y|^\alpha} \to 0$ as $x \to y$ for all $y \in [0,1]$. Hence $f \in \lambda_\alpha([0,1])$ and $\lambda_\alpha([0,1])$ is closed.

We now show that $\lambda_\alpha([0,1])$ is infinite dimensional. Consider $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ for $n \in \mathbb{N}$. Clearly the f_n are independent, so to complete the proof of infinite dimensionality we need to show that $f_n \in \lambda_\alpha([0,1])$ for all $n \in \mathbb{N}$. Indeed,

$$\frac{|x^n - y^n|}{|x-y|^\alpha} = \frac{|(x-y)\left(\sum_{k=0}^{n-1} x^k y^{n-k-1}\right)|}{|x-y|^\alpha} = |x-y|^{1-\alpha} \left|\sum_{k=0}^{n-1} x^k y^{n-k-1}\right| \to 0,$$

as $x \to y$ for all $y \in [0,1]$.

If $\alpha = 1$, we have $f \in \Lambda_\alpha([0,1])$ if and only if for all $y \in [0,1]$, $\lim_{x\to y} \frac{|f(x)-f(y)|}{|x-y|} = 0$, which happens if and only if $f'(x) = 0$ for all $x \in [0,1]$, which happens if and only if f is constant.

Problem 5.19

Let X be an infinite-dimensional normed vector space.

a. *There is a sequence $\{x_j\}$ in X such that $||x_j|| = 1$ for all j and $||x_j - x_k|| \geq 1/2$ for $j \neq k$. (Construct x_j inductively, using Exercises 12b and 18.)*

b. *X is not locally compact.*

Solution:

a. Choose any vector $x_1 \in X$ such that $||x_1|| = 1$. Then $M_1 = \mathbb{C}x_1$ is a closed proper subspace by Exercise 18. Invoking Exercise 12b with $\epsilon = 1/2$, we can find $x_2 \in X$ such that $||x_2|| = 1$ and $\inf\{||m + x_2|| : m \in M\} \geq 1/2$. In particular, $||x_2 - x_1|| \geq 1/2$.

We can now proceed inductively. Suppose there are $\{x_1, \ldots, x_k\}$ vectors such that $||x_j|| = 1$ for all $1 \leq j \leq k$ and $||x_j - x_k|| \geq 1/2$ for $j \neq k$. Then $M_k = \text{span}\{x_1, \ldots, x_k\}$ is a closed proper subspace of X. From Exercise 12b, there exists a $x_{k+1} \in X$ such that $||x_{k+1}|| = 1$ and $\inf\{||m + x_{k+1}|| : m \in M\} \geq 1/2$. In particular, $||x_{k+1} - x_j|| \geq 1/2$ for $1 \leq j \leq k$. The sequence can therefore be constructed by induction.

b. Suppose X is locally compact. Then there exists a compact set K containing the origin and a $\delta > 0$ such that $U = \{x \in X : ||x|| < \delta\}$ is contained in K. Take the sequence $\{x_j\}$ constructed in part (a), and consider the rescaled sequence $\{y_j\} = \{(1/2)\delta x_j\}$. Then $\{y_j\}$ is a sequence contained in K, and hence there must be a convergent subsequence. However, $||y_i - y_k|| \geq (1/4)\delta$ for all $i \neq k$, so no subsequence can be Cauchy. This contradiction completes the proof.

Problem 5.22

Suppose that X and Y are normed vector spaces and $T \in L(X, Y)$.

a. *Define $T^\dagger : Y^* \to X^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in L(Y^*, X^*)$ and $||T^\dagger|| = ||T||$. $T^\dagger$ is called the adjoint or transpose of T.*

b. *Applying the construction in (a) twice, one obtains $T^{\dagger\dagger} \in L(X^{**}, Y^{**})$. If X and Y are identified with their natural images $\hat{X}$ and $\hat{Y}$ in X^{**} and Y^{**}, then $T^{\dagger\dagger}|X = T$.*

c. *$T^\dagger$ is injective iff the range of T is dense in Y.*

d. *If the range of $T^\dagger$ is dense in X^*, then T is injective; the converse is true if X is reflexive.*

Solution:

a. We first notice that $T^\dagger f \in X^*$ for all $f \in Y^*$, since $T^\dagger f = f \circ T$ is the product of two bounded linear operators, and hence is itself a bounded linear operator. Next, we show $T^\dagger \in L(Y^*, X^*)$:

$$T^\dagger(cf) = (cf) \circ T = c(f \circ T) = cT^\dagger f.$$

$$T^\dagger(f + g) = (f + g) \circ T = f \circ T + g \circ T = T^\dagger f + T^\dagger g.$$

To see that $T^\dagger$ is bounded, take $f \in L(Y, \mathbb{C})$, and since $T \in L(X,Y)$, using a property of the product of two bounded linear operators we obtain

$$||T^\dagger(f)|| = ||f \circ T|| \leq ||T||\ ||f||.$$

Therefore $T^\dagger \in L(Y^*, X^*)$, and since the operator norm of $T^\dagger$ is the infimum of all such bounds, $||T^\dagger|| \leq ||T||$.

We now show $||T|| \leq ||T^\dagger||$ by proving $||Tx|| \leq ||T^\dagger||\ ||x||$ for all $x \in X$. If $Tx = 0$, the inequality holds trivially, so suppose $Tx \neq 0$. Then by Theorem 5.8, there exists a $f \in Y^*$ such that $||f|| = 1$ and $(T^\dagger f)(x) = f(Tx) = ||Tx||$.

$$||Tx|| = ||(T^\dagger f)(x)|| \leq ||T^\dagger f||\ ||x|| \leq ||T^\dagger||\ ||f||\ ||x|| = ||T^\dagger||\ ||x||.$$

This shows that $||T^\dagger|| = ||T||$.

b. First, we show that $T^{\dagger\dagger}\alpha \in Y^{**}$ for all $\alpha \in X^{**}$. Indeed, $T^{\dagger\dagger}\alpha = \alpha \circ T^\dagger$ is the product of two bounded linear operators, hence $T^{\dagger\dagger}\alpha \in Y^{**}$.

Next, we show that $T^{\dagger\dagger}$ is a bounded linear operator:

$$T^{\dagger\dagger}(c\alpha) = (c\alpha) \circ T^\dagger = cT^{\dagger\dagger}(\alpha).$$
$$T^{\dagger\dagger}(\alpha + \beta) = (\alpha + \beta) \circ T^\dagger = \alpha \circ T^\dagger + \beta \circ T^\dagger = T^{\dagger\dagger}(\alpha) + T^{\dagger\dagger}(\beta).$$
$$||T^{\dagger\dagger}(\alpha)|| = ||\alpha \circ T^\dagger|| \leq ||T^\dagger||\ ||\alpha||.$$

It will now be shown that $T^{\dagger\dagger}|X = T$ after identifying X, Y with $\hat{X}, \hat{Y}$. For any $f \in Y^*$, we have

$$(T^{\dagger\dagger}\hat{x})(f) = \hat{x} \circ T^\dagger f = \hat{x} \circ (f \circ T) = f \circ T(x) = \hat{T(x)} f.$$

Hence $T^{\dagger\dagger}\hat{x} = \hat{T(x)}$.

c. Suppose that the range of T is not dense. Then there exists a non-empty open set $U \subseteq Y$ such that $U \cap \overline{\text{Range}(T)} = \emptyset$. Since $\overline{\text{Range}(T)}$ is a closed subspace and there exists a $y \in U$, by Theorem 5.8a, there exists $f \in Y^*$ such that $f(y) \neq 0$ and $f|\overline{\text{Range}(T)} = 0$. In other words, $T^\dagger f = 0$ but $f \neq 0$, and hence $T^\dagger$ is not injective.

Conversely, suppose that the range of T is dense in Y. If there exists a $f \in Y^*$ such that $T^\dagger f = 0$, then $f \circ T(x) = 0$ for all $x \in X$. Hence f is zero on a dense subset of Y, and by continuity of f it follows that f is identically zero on Y. (This is known fact from topology, also stated in Exercise 4.16b and proved on the 564 final exam.) Therefore, $T^\dagger$ is injective.

d. Suppose the range of $T^\dagger$ is dense in X^*. If $T(x) = 0$ for some $x \in X$, then consider the associated linear functional $\hat{x} \in X^{**}$. For any $f \in Y^*$, we have

$$\hat{x}(T^\dagger f) = \hat{x}(f \circ T) = f \circ T(x) = 0.$$

Therefore, $\hat{x}$ is zero on the range of $T^\dagger$, and hence by continuity, $\hat{x}$ is identically zero on all of X^*. Therefore $x = 0$, and T is injective.

On the other hand, suppose the range of $T^\dagger$ is not dense in X^* and X is reflexive. There exists a non-empty open set $U \subseteq X^*$ such that $U \cap \overline{\text{range}(T^\dagger)} = \emptyset$. By Theorem 5.8a, there exists a non-zero $\alpha \in X^{**}$ such that $\alpha|\overline{\text{range}(T^\dagger)} = 0$. Since X is reflexive, there exist a $x \neq 0 \in X$ such that $\alpha = \hat{x}$.

Suppose that $Tx \neq 0$. Then by Theorem 5.8b, there exists a $f \in Y^*$ such that $f \circ Tx = ||Tx||$. This leads to the following contradiction:

$$0 = \hat{x}(T^\dagger f) = \hat{x}(f \circ T) = f \circ T(x) = ||T(x)||.$$

Therefore, we have found a non-zero $x \in X$ such that $Tx = 0$. Hence T is not injective.

Problem 5.31

Let X, Y be Banach spaces and let $S : X \to Y$ be an unbounded linear map (for the existence of which, see §5.6*). Let $\Gamma(S)$ be the graph of S, a subspace of $X \times Y$.*

a. *$\Gamma(S)$ is not complete.*

b. *$\Gamma(S)$ Define $T : X \to \Gamma(S)$ by $Tx = (x, Sx)$. Then T is closed but not bounded.*

c. *$T^{-1} : \Gamma(S) \to X$ is bounded and surjective but not open.*

Solution:

a. Suppose $\Gamma(S)$ is complete. Since $X \times Y$ is a metric space, a subset F is closed if and only if the limit of every convergent sequence in F belongs to F. Any convergent sequence in $\Gamma(S)$ is Cauchy, and hence its limit is in $\Gamma(S)$ by completeness. Therefore, $\Gamma(S)$ is closed. By the Closed Graph Theorem, S is bounded. This is a contradiction, hence $\Gamma(S)$ is not complete.

b. First, we show T is not bounded. Choose any $C > 0$. Then there exists an $x \in X$ such that $||Sx|| > C||x||$ since S is unbounded. Using the definition of the product norm, we obtain

$$||Tx|| = \max\{||x||, ||Sx||\} > C||x||.$$

Hence T is unbounded. Next we show that $\Gamma(T) \subseteq X \times \Gamma(S)$ is closed. Suppose a sequence $\{(x_n, (x_n, Sx_n))\} \in \Gamma(T)$ converges to an element $(x, (\tilde{x}, S\tilde{x})) \in X \times \Gamma(S)$. Then

$$||x_n - x|| \leq \max\{||x_n - x||, ||(x_n, Sx_n) - (\tilde{x}, S\tilde{x})||\} = ||(x_n, (x_n, Sx_n)) - (x, (\tilde{x}, S\tilde{x}))|| \to 0$$

as $n \to \infty$, and $\lim x_n = x$. Similarly,

$$\begin{aligned}
||x_n - \tilde{x}|| &\leq \max\{||x_n - \tilde{x}||, ||Sx_n - S\tilde{x})|| \\
&= ||(x_n - \tilde{x}, Sx_n - S\tilde{x})|| \\
&= ||(x_n, Sx_n) - (\tilde{x}, S\tilde{x})|| \\
&\leq \max\{||x_n - x||, ||(x_n, Sx_n) - (\tilde{x}, S\tilde{x})||\} \\
&= ||(x_n, (x_n, Sx_n)) - (x, (\tilde{x}, S\tilde{x}))|| \to 0
\end{aligned}$$

as $n \to \infty$, and $\lim x_n = \tilde{x}$. Therefore, $\tilde{x} = x$ and $(x, (\tilde{x}, S\tilde{x})) = (x, (x, Sx)) \in \Gamma(T)$. This shows that T is closed.

c. We have $T^{-1} : \Gamma(S) \to X$ defined as $T^{-1}((x, Sx)) = x$. It is clear that $T \circ T^{-1} = T^{-1} \circ T = I$ and hence T^{-1} is surjective. To see that T^{-1} is bounded, notice

$$||T^{-1}((x, Sx))|| = ||x|| \leq \max\{||x||, ||Sx||\} = ||(x, Sx)||,$$

for all $(x, Sx) \in \Gamma(S)$.

Lastly, if T^{-1} was open, then T would be continuous. However, as shown previously, T is unbounded, hence T^{-1} is not open.

Problem 5.42

Let E_n be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on f) such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$.

a. *E_n is nowhere dense in $C([0,1])$. (Any real $f \in C([0,1])$ can be uniformly approximated by a piecewise linear function g whose linear pieces, finite in number, have slope $\pm 2n$. If $||h - g||_u$ is sufficiently small, then $h \notin E_n$.)*

b. *The set of nowhere differentiable functions is residual in $C([0,1])$.*

Solution:

a. First, we show that any real $f \in C([0,1])$ can be uniformly approximated by a piecewise linear function ψ.

Let $\epsilon > 0$. Since f is uniformly continuous on $[0,1]$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/2$ when $|x - y| < \delta$. Choose $N \in \mathbb{N}$ such that $1/N < \delta$. Let $x_i = i/N$, where i is an integer such that $0 \leq i \leq N$. Define $\psi \in C([0,1])$ such that $\psi(x_i) = f(x_i)$ and ψ is linear on $[x_i, x_{i+1}]$.

Now take $x \in [0,1]$, supposing $x \in [x_i, x_{i+1}]$. Then

$$\begin{aligned}
|\psi(x) - f(x)| &\leq |\psi(x) - \psi(x_i)| + |\psi(x_i) - f(x)| \\
&\leq |\psi(x_{i+1}) - \psi(x_i)| + |\psi(x_i) - f(x)| \\
&= |f(x_{i+1}) - f(x_i)| + |f(x_i) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Now, since f can be uniformly approximated by a piecewise linear function ψ, it is easy to see that we can uniformly approximate ψ by a function g whose linear pieces, finite in number, have slope of absolute value greater than $2n$. Indeed, if one of the linear pieces of ψ has slope of absolute value less than $2n$, approximate it by a see-saw function whose right-hand derivative has absolute value of $2n$.

We show that each E_n is closed. Suppose (f_k) is a sequence in E_n, and $f_k \to f$ in $C([0,1])$. Then for each f_k, there exists a $x_k \in [0,1]$ such that $|f_k(x) - f_k(x_k)| \leq n|x - x_k|$. We obtain a bounded sequence (x_k) in $[0,1]$, which must have a convergent subsequence. Denote the limit of this subsequence as x_0. Let $\epsilon > 0$. Choose $m \in \mathbb{N}$ such that $||f - f_m||_u < \epsilon/2$ and $|f_m(x) - f_m(x_0)| \leq n|x - x_0|$. Then

$$\begin{aligned}
|f(x) - f(x_0)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
&\leq 2||f - f_m||_u + |f_m(x) - f_m(x_0)| \\
&\leq \epsilon + n|x - x_0|.
\end{aligned}$$

Since this holds for all $\epsilon > 0$, $f \in E_n$ and hence $\overline{E_n} = E_n$. We now show that E_n is nowhere dense in $C([0,1])$. If $f \in E_n$, suppose there exists a ball of radius $\epsilon > 0$ centered at f contained in E_n. Take g as above, ie, a piecewise linear function such that $||f - g|| < \epsilon$ whose finitely many linear pieces have slope of absolute value greater than $2n$. Then for any $x_0 \in [0,1]$, there exists a y sufficiently close to x_0 such that

$$\frac{|g(y) - g(x_0)|}{|y - x_0|} \geq 2n.$$

Therefore, $g \notin E_n$. Hence there is no open ball centered at f contained in $\overline{E_n} = E_n$, so E_n is nowhere dense in $C([0,1])$.

b. Let A denote the set of functions in $C([0,1])$ that are nowhere differentiable. We show that $A^c \subseteq \cup E_n$. If $f \in C([0,1])$ is differentiable at some $x_0 \in [0,1]$, then there exists $\delta, M > 0$ such that if $|x - x_0| < \delta$, then

$$\frac{|f(x_0) - f(x)|}{|x_0 - x|} \leq M.$$

On the other hand, if $|x - x_0| > \delta$, we see that

$$\frac{|f(x_0) - f(x)|}{|x_0 - x|} \leq \frac{2||f||_u}{\delta}.$$

It follows that if $n \in \mathbb{N}$ is such that $n > M$ and $n > 2||f||_u/\delta$, then $f \in E_n$. Hence

$$A^c \subseteq \bigcup E_n,$$

and thus, A^c is the subset of a meager set and is hence also meager. Therefore, A is the complement of a meager set. This shows that the set of nowhere differentiable functions in residual in $C([0,1])$.

Problem 5.48

Suppose that X is a Banach space.

a. *The norm-closed unit ball $B = \{x \in X : ||x|| \le 1\}$ is also weakly closed. (Use Theorem 5.8d.)*

b. *If $E \subset X$ is bounded (with respect to the norm), so is its weak closure.*

c. *If $F \subset X^*$ is bounded (with respect to the norm), so is its weak* closure.*

d. *Every weak*-Cauchy sequence in X^* converges. (Use Exercise 38.)*

Solution:

a. Let $\langle x_\alpha \rangle$ be a net in B that converges weakly to $x \in X$. By Theorem 5.8d, $||x_0|| = ||\hat{x}_0||$ for all $x_0 \in X$. Hence for every x_α in the net, we have

$$\hat{x}_\alpha \in B^{**} = \{\alpha \in X^{**} : ||\alpha|| \le 1\}.$$

Since $\hat{x_\alpha}(f) = f(x_\alpha) \to f(x) = \hat{x}(f)$ for all $f \in X^*$, we see that the net $\langle \hat{x}_\alpha \rangle$ converges to $\hat{x}$ in the weak* topology on $(X^*)^*$. Since the weak* topology is Hausdorff, any compact set is closed, and hence by Alaoglu's Theorem, $\hat{x} \in B^{**}$. It follows that $||x|| = ||\hat{x}|| \le 1$, and $x \in B$. Therefore B is weakly closed.

b. If $E \subseteq X$ is bounded, then there exists a $C > 0$ such that $||x|| \le C$ for all $x \in E$. Then $||(1/C)x|| \le 1$ for all $x \in E$. Let $\langle x_\alpha \rangle$ be a net in E that converges weakly to $x \in X$. By continuity of scalar multiplication, $\langle (1/C)x_\alpha \rangle$ converges weakly to $(1/C)x$, and by part (a), $(1/C)x \in B$. Hence $||x|| \le C$. Therefore, the weak closure of E is bounded.

c. If $F \subseteq X^*$ is bounded, then there exists a $C > 0$ such that $||f|| \le C$ for all $f \in X^*$. Let $\langle f_\alpha \rangle$ be a net in F that converges to $f \in X^*$ in the weak* topology. By continuity of scalar multiplication, the net $\langle (1/C)f_\alpha \rangle$ converges to $(1/C)f$ in the weak* topology. Since $\langle (1/C)f_\alpha \rangle$ is a net in $B^* = \{f \in X^* : ||f|| \le 1\}$, by Alaoglu's Theorem, $(1/C)f$ is also in B^*. Hence $||f|| \le C$, and therefore the weak* closure of F is bounded.

d. Let (f_n) be a Cauchy sequence in X^* with respect to the weak* topology. Then for all $x \in X$, $|f_n(x) - f_m(x)| \to 0$ as $n, m \to \infty$. By completeness of $\mathbb{C}$, $\lim f_n(x)$ exists for all $x \in X$. By Exercise 38, if we define $f(x) = \lim f_n(x)$, then $f \in X^*$. It is clear that $f_n \to f$ in the weak* topology.

Problem 5.57

Suppose that $\mathcal{H}$ is a Hilbert space and $T \in L(\mathcal{H}, \mathcal{H})$.

a. *There is a unique $T^* \in L(\mathcal{H}, \mathcal{H})$, called the adjoint of T, such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathcal{H}$. (Cf. Exercise 22. We have $T^* = V^{-1}T^\dagger V$ where V is the conjugate-linear isomorphism from $\mathcal{H}$ to $\mathcal{H}^*$ in Theorem 5.25, $(Vy)(x) = \langle x, y \rangle$.)*

b. *$||T^*|| = ||T||$, $||T^*T|| = ||T||^2$, $(aS + bT)^* = \overline{a}S^* + \overline{B}T^*$, $(ST)^* = T^*S^*$, and $T^{**} = T$.*

c. *Let $\mathcal{R}$ and $\mathcal{N}$ denote range and nullspace; then $\mathcal{R}(T)^\perp = \mathcal{N}(T^*)$ and $\mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}$.*

d. *T is unitary iff T is invertible and $T^{-1} = T^*$.*

Solution:

a. Define $T^* = V^{-1}T^\dagger V$. Then T^* is the composition of bounded linear operators, hence T^* is a bounded linear operator. For all $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\langle x, T^*y\rangle &= (VT^*y)(x)\\ &= (VV^{-1}T^\dagger Vy)(x)\\ &= (T^\dagger Vy)(x)\\ &= ((Vy)\circ T)(x)\\ &= (Vy)(Tx)\\ &= \langle Tx, y\rangle.\end{aligned}$$

To show uniqueness, suppose there exists $S \in L(\mathcal{H}, \mathcal{H})$ such that $\langle Tx, y\rangle = \langle x, Sx\rangle$. Then $\langle x, T^*y\rangle = \langle x, Sy\rangle$, hence $\langle x, (T^* - S)y\rangle = 0$ for all $x, y \in \mathcal{H}$. In particular, $||(T^* - S)y||^2 = \langle (T^* - S)y, (T^* - S)y\rangle = 0$ for all $y \in \mathcal{H}$, hence $T^* - S = 0$.

b. First, we show $T^{**} = T$. For all $x, y \in \mathcal{H}$,

$$\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle.$$

Next, we show $||T^*|| = ||T||$. We know $||T|| = ||T^\dagger||$ from Exercise 22, and we know $||V|| = ||V^{-1}|| = 1$ since V is an isometry and an isomorphism. Hence,

$$||T^*x|| = ||V^{-1}T^\dagger Vx|| \leq ||V^{-1}||\ ||T^\dagger||\ ||V||\ ||x|| = ||T||\ ||x||.$$

Thus we have shown $||T^*|| \leq ||T||$. Combining this with the fact that $T = T^{**}$, we obtain $||T|| = ||(T^*)^*|| \leq ||T^*||$.

Next, we show $||T^*T|| = ||T||^2$. For any $x \in \mathcal{H}$,

$$||T^*Tx|| \leq ||T||\ ||T^*||\ ||x|| = ||T||^2\ ||x||,$$

so $||T^*T|| \leq ||T||^2$. On the other hand,

$$||Tx||^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = (VT^*Tx)(x) \leq ||VT^*Tx||\ ||x|| \leq ||T^*T||\ ||x||^2.$$

Therefore, $||T|| \leq ||T^*T||^{1/2}$.

To see $(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$, notice that for all $x, y \in \mathcal{H}$,

$$\langle (aS + bT)x, y\rangle = a\langle Sx, y\rangle + b\langle Tx, y\rangle = a\langle x, S^*y\rangle + b\langle x, T^*y\rangle = \langle x, \overline{a}S^*y + \overline{b}T^*y\rangle = \langle x, (\overline{a}S^* + \overline{b}T^*)y\rangle.$$

To see $(ST)^* = T^*S^*$, notice that for all $x, y \in \mathcal{H}$,

$$\langle STx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle.$$

c. Let $y \in R(T)^\perp$. Then $\langle y, Tx \rangle = 0$ for all $x \in \mathcal{H}$. Therefore $\langle T^*y, x \rangle = 0$ for all $x \in \mathcal{H}$, and in particular, $||T^*y||^2 = \langle T^*y, T^*y \rangle = 0$. Therefore $y \in \mathcal{N}(T^*)$. On the other hand, let $x \in \mathcal{N}(T^*)$. Then $\langle T^*x, y \rangle = 0$ for all $y \in \mathcal{H}$. Hence $\langle x, Ty \rangle = 0$ for all $y \in \mathcal{H}$, and $x \in R(T)^\perp$. Therefore, $R(T)^\perp = \mathcal{N}(T^*)$.

Suppose $x \in \mathcal{N}(T)^\perp$. Then $\langle x, v \rangle = 0$ for all v such that $Tv = 0$. Let $y \in R(T^*)^\perp$. Then $\langle Ty, w \rangle = \langle y, T^*w \rangle = 0$ for all $w \in \mathcal{H}$, hence $||Ty||^2 = \langle Ty, Ty \rangle = 0$. Therefore $Ty = 0$, so $\langle x, y \rangle = 0$. It follows that $x \in (R(T^*)^\perp)^\perp$. By Exercise 56, $(R(T^*)^\perp)^\perp = \overline{R(T^*)}$, hence $x \in \overline{R(T^*)}$. Therefore $\mathcal{N}(T)^\perp \subseteq \overline{R(T^*)}$. On the other hand, let $y \in \overline{R(T^*)}$. Then there exists a sequence (T^*x_n) that converges to y. If $Tz = 0$ for some $z \in \mathcal{H}$, then

$$\langle y, z \rangle = \langle \lim T^*x_n, z \rangle = \lim \langle T^*x_n, z \rangle = \lim \langle x_n, Tz \rangle = 0.$$

Therefore $y \in \mathcal{N}(T)^\perp$.

d. Suppose T is unitary. Then T is invertible by definition, and for all $x, y \in \mathcal{H}$,

$$\langle Tx, y \rangle = \langle Tx, TT^{-1}y \rangle = \langle x, T^{-1}y \rangle.$$

It follows that $T^{-1} = T^*$. On the other hand, if T is invertible and $T^{-1} = T^*$, then for all $x, y \in \mathcal{H}$,

$$\langle Tx, Ty \rangle = \langle x, T^{-1}Ty \rangle = \langle x, y \rangle.$$

Problem 5.59
Every closed convex set K in a Hilbert space has a unique element of minimal norm. (If $0 \in K$, the result is trivial; otherwise, adapt the proof of Theorem 5.24.)

Solution:
Let $\delta = \inf\{||x|| : x \in K\}$, and let $\{x_n\}$ be a sequence in K such that $||x_n|| \to \delta$. By the paralellogram law,

$$||x_n - x_m||^2 = 2||x_n||^2 + 2||x_m||^2 - ||x_n + x_m||^2.$$

By convexity, $(1/2)(x_n + x_m) \in K$. Therefore $||(1/2)(x_n + x_m)|| \geq \delta$, hence

$$||x_n - x_m||^2 \leq 2||x_n||^2 + 2||x_m||^2 - 4\delta^2.$$

As $m, n \to \infty$, this quantity goes to zero, hence $\{x_n\}$ is Cauchy. Let $x = \lim x_n$. Then $x \in K$ since K is closed and $||x|| = \delta$, hence x is an element of minimal norm.

To show x is unique, suppose $y \in K$ is also such that $||y|| = \delta$. Again, using the paralellogram law and the fact that $(1/2)(x + y) \in K$,

$$||y-x||^2 = 2||x||^2 + 2||y||^2 - ||x+y||^2 \leq 2\delta^2 + 2\delta^2 - 4\delta^2 = 0.$$

Hence $y = x$.

Problem 5.64

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space with orthonormal basis $\{u_n\}_1^\infty$.

a. *For $k \in \mathbb{N}$, define $L_k \in L(\mathcal{H}, \mathcal{H})$ by $L_k(\sum_1^\infty a_n u_n) = \sum_k^\infty a_n u_{n-k}$. Then $L_k \to 0$ in the strong operator topology but not in the norm topology.*

b. *For $k \in \mathbb{N}$, define $R_k \in L(\mathcal{H}, \mathcal{H})$ by $R_k(\sum_1^\infty a_n u_n) = \sum_1^\infty a_n u_{n+k}$. Then $R_k \to 0$ in the weak operator topology but not in the strong operator topology.*

c. *$R_k L_k \to 0$ in the strong operator topology, but $L_k R_k = I$ for all k. (Use Exercise 53b.)*

Solution:

a. There is some confusion in the statement of the question since u_0 is undefined, however it appears after applying L_k. We proceed assuming $u_0 = 0$. Fix $x \in \mathcal{H}$. Then $x = \sum a_n u_n$ for some coefficients a_n. Applying L_k to x, we see that

$$||L_k x||^2 = ||\sum_{n=k}^{\infty} a_n u_{n-k}||^2 = \sum_{n=k+1}^{\infty} |a_n|^2 \to 0$$

as $k \to \infty$. Hence $L_k \to 0$ in the strong operator topology.

However, $||L_k|| \geq 1$ for all k. We can consider the vector u_{k+1}, and since $||u_{k+1}|| = 1$,

$$||L_k|| \geq ||L_k u_{k+1}|| = 1.$$

Therefore L_k does not converge to zero in the norm topology.

b. Let $f \in \mathcal{H}^*$. Then there exists a $y = \sum b_n u_n \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$. For any $x = \sum a_n u_n \in \mathcal{H}$, after some manipulations and applying the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\begin{aligned}
\big|f(R_k(x))\big|^2 = \big|f\big(\sum_{i=1}^{\infty} a_i u_{i+k}\big)\big|^2 &= \big|\langle \sum_{i=1}^{\infty} a_i u_{i+k}, \sum_{j=1}^{\infty} b_j u_j \rangle\big|^2 \\
&= \big|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle a_i u_{i+k}, b_j u_j \rangle\big|^2 \\
&= \big|\sum_{i=1}^{\infty} \langle a_i u_{i+k}, b_{i+k} u_{i+k} \rangle\big|^2 \\
&= \big|\sum_{i=1}^{\infty} a_i b_{i+k}\big|^2 \\
&\leq \big(\sum_{i=1}^{\infty} |a_i|^2\big)\ \big(\sum_{j=1+k}^{\infty} |b_j|^2\big).
\end{aligned}$$

Since $\sum_{i=1}^{\infty} |a_i|^2$ is finite and $\sum_{j=1+k}^{\infty} |b_j|^2 \to 0$ as $k \to \infty$, we have $f(R_k(x)) \to 0$ as $k \to 0$, hence $R_k \to 0$ in the weak operator topology.

On the other hand, $||R_k(u_1)|| = ||u_{1+k}|| = 1$ for all k. Hence R_k does not converge to zero in the strong operator topology.

c. Fix $x \in \mathcal{H}$. Then $x = \sum a_n u_n$ for some coefficients a_n. Applying $R_k L_k$ to x, we see that

$$\begin{aligned} ||R_k L_k x||^2 = ||R_k(\sum_{n=k}^{\infty} a_n u_{n-k})||^2 &= ||R_k(\sum_{i=1}^{\infty} a_{k+i} u_i)||^2 \\ &= ||\sum_{i=1}^{\infty} a_{k+i} u_{k+i}||^2 \\ &= ||\sum_{n=k+1}^{\infty} a_n u_n||^2 \\ &= \sum_{n=k+1}^{\infty} |a_n|^2. \end{aligned}$$

Hence $||R_k L_k x||^2 \to 0$ as $k \to 0$. Therefore, $R_k L_K \to 0$ in the strong operator topology.

On the other hand, if we apply $L_k R_k$ on $x = \sum a_n u_n$, we obtain

$$\begin{aligned} L_k R_k x &= L_k(\sum_{n=1}^{\infty} a_n u_{n+k}) \\ &= L_k(\sum_{m=1+k}^{\infty} a_{m-k} u_m) \\ &= \sum_{m=1+k}^{\infty} a_{m-k} u_{m-k} \\ &= \sum_{n=1}^{\infty} a_n u_n = x. \end{aligned}$$

Hence $L_k R_k = I$.