Folland: Real Analysis, Chapter 6
Sébastien Picard

Problem 6.8
Suppose $\mu(X) = 1$ and $f \in L^p$ for some $p > 0$, so that $f \in L^q$ for $0 < q < p$.
a. $\log ||f||_q \geq \int \log |f|$. *(Use Exercise 42d in §3.5, with $F(t) = e^t$).*
b. $(\int |f|^q - 1)/q \geq \log ||f||_q$, *and* $(\int |f|^q - 1)/q \to \int \log |f|$ *as* $q \to 0$.
c. $\lim_{q\to 0} ||f||_q = \exp(\int \log |f|)$.

Solution:
(a) Define $g = q \log |f|$.

Case 1: Suppose $g \notin L^1$. Then $\int |\log |f|^q| = \infty$. We can split up the integral

$$\int |\log |f|^q| = \int \log |f|^q \chi_{|f|>1} - \int \log |f|^q \chi_{|f|<1} \leq \int |f|^q - \int \log |f|^q \chi_{|f|<1}.$$

Since $f \in L^q$, we must have $\int \log |f|^q \chi_{|f|<1} = -\infty$. Therefore, since $\int \log |f|^q \chi_{|f|>1}$ is finite, $\int q \log |f| = -\infty$ and the inequality holds trivially.

Case 2: Suppose $g \in L^1$. Let $F(t) = e^t$. We know that $F(t)$ is convex on the real line since the exponential function is absolutely continuous on every compact interval and $F''(t) = F(t) > 0$. Thus we can apply Jensen's inequality:

$$\begin{aligned}\int \log |f| &= \frac{1}{q} \int \log |f|^q \\ &= \frac{1}{q} \int g \\ &= \frac{1}{q} \log e^{\int g} \\ &\leq \frac{1}{q} \log \int e^g \\ &= \frac{1}{q} \log \int |f|^q = \log ||f||_q.\end{aligned}$$

(b)
The first inequality follows directly from the fact that $\log(x) \leq x - 1$.

$$\log ||f||_q = \frac{1}{q} \log \int |f|^q \leq \frac{1}{q}(\int |f|^q - 1).$$

In order to prove the limit, we will need the fact that the map $h : (0, \infty) \to \mathbb{R}$ defined by $h(x) = (a^x - 1)/x$ is a monotone increasing function for any $a \geq 0$. If $a = 0$, this is immediate. If $a > 0$, we prove monotonicity by showing that $h'(x) \geq 0$. Indeed,

$$h'(x) = \frac{a^x \log a^x - a^x + 1}{x^2},$$

hence the result will follow if we show that $w : (0, \infty) \to \mathbb{R}$, $w(x) = x \log x - x$ is greater than or equal to -1. Since w' only vanishes at $x = 1$, and $w''(1) > 0$, we see that the minimum of w occurs at $x = 1$ and hence $w \geq -1$.

We now prove the limit. By l'Hopital's rule, pointwise we have

$$\lim_{q \to 0} \frac{|f|^q - 1}{q} = \lim_{q \to 0} \frac{\log |f| \cdot |f|^q}{1} = \log |f|.$$

As shown before, $(|f|^q - 1)/q$ is monotone increasing, hence the limit as $q \to 0$ is monotone decreasing. By the Monotone Convergence Theorem, we have

$$\lim_{q \to 0} \int \frac{1}{q}(|f|^q - 1) = \int \lim_{q \to 0} \frac{1}{q}(|f|^q - 1) = \int \log |f|.$$

(c) From part (a), after composing both sides with the exponential function, we see that $||f||_q \geq \exp(\int \log |f|)$ for all $0 < q < p$. From part (b), we see that

$$\limsup_{q \to 0} \ \log ||f||_q \leq \limsup_{q \to 0} \ \Big(\int |f|^q - 1 \Big)/q \to \int \log |f|.$$

Therefore, $\limsup_{q \to 0} ||f||_q \leq \exp(\int \log |f|)$. Therefore, the limit exists, and $\lim_{q \to 0} ||f||_p = \exp(\int \log |f|)$.

Problem 6.10

Suppose $1 \leq p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $||f_n - f||_p \to 0$ iff $||f_n||_p \to ||f||_p$. (Use Exercise 20 in §2.3.)

Solution:

Suppose $||f_n - f||_p \to 0$. Then by the reverse triangle inequality,

$$| \ ||f_n||_p - ||f||_p \ | \leq ||f_n - f||_p \to 0$$

as $n \to \infty$.

On the other hand, suppose $||f_n||_p \to ||f||_p$. Notice

$$|f_n - f|^p \leq 2^p(|f|^p + |f_n|^p).$$

We can define $g_n = 2^p(|f|^p + |f_n|^p)$ and $g = 2^{p+1}|f|^p \in L^1$. Then $g_n \to g$ a.e, and furthermore,

$$\lim \int g_n = 2^p \int |f|^p + \lim \left(2^p \int |f_n|^p\right) = \int 2^{p+1}|f|^p = \int g.$$

By Exercise 20 in §2.3,

$$\lim \int |f_n - f|^p = \int \lim |f_n - f|^p = 0.$$

Problem 6.20

Suppose $\sup_n ||f_n||_p < \infty$ *and* $f_n \to f$ *a.e.*

a. *If* $1 < p < \infty$*, then* $f_n \to f$ *weakly in* L^p*. (Given* $g \in L^q$*, where* q *is conjugate to* p*, and* $\epsilon > 0$*, there exists (i)* $\delta > 0$ *such that* $\int_E |g|^q < \epsilon$ *whenever* $\mu(E) < \delta$*, (ii)* $A \subset X$ *such that* $\mu(A) < \infty$ *and* $\int_{X\setminus A} |g|^q < \epsilon$*, and (iii)* $B \subset A$ *such that* $\mu(A\setminus B) < \delta$ *and* $f_n \to f$ *uniformly on* B*.)*

b. *The result of (a) is false in general for* $p = 1$*. (Find counterexamples in* $L^1(\mathbb{R}, m)$ *and* l^1*.) It is, however, true for* $p = \infty$ *if* μ *is* $\sigma-$*finite and weak convergence is replaced by weak* convergence.*

Solution:

a. Claim (i) follows from Corollary 3.6, and claim (iii) is Egoroff's theorem (2.33). We first prove claim (ii). By Proposition 2.20, $P = \{x : |g|^q > 0\}$ is $\sigma-$finite. Hence we can assume that $P = \cup_i^\infty P_i$ where $\mu(P_i) < \infty$ and P_i are disjoint. Therefore,

$$\int_X |g|^q = \int_P |g|^q = \sum_{i=1}^{\infty} \int_{P_i} |g|^q < \infty.$$

Since the sequence converges, there exists an $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} \int_{P_i} |g|^q < \epsilon$. Therefore, $A = \cup_{i=1}^N P_i$ has the required properties, which proves (ii).

We now prove that $f_n \to f$ weakly in L^p. Take $g \in L^q$ and any $\epsilon > 0$. Since $\sup_n ||f_n||_p \le C_0 < \infty$, we have that $f \in L^p$ by Fatou's lemma:

$$\int |f|^p \le \liminf \int |f_n|^p \le C_0^p < \infty.$$

By (i), choose $\delta > 0$ such that

$$\int_E |g|^q < \left(\frac{\epsilon}{6C_0}\right)^q,$$

when $\mu(E) < \delta$. By (ii), choose $A \subset X$ such that $\mu(A) < \infty$ and

$$\int_{X\setminus A} |g|^q < \left(\frac{\epsilon}{6C_0}\right)^q.$$

By (iii), choose $B \subset A$ and $N \in \mathbb{N}$ such that $\mu(A\setminus B) < \delta$ and

$$|f_n(x) - f(x)| < \frac{\epsilon}{3\mu(B)^{1/p}||g||_q},$$

for all $x \in B$ and $n \geq N$.

With this setup, when $n \geq N$, we apply Holder's inequality to obtain

$$\begin{aligned}
\left|\int_X f_n g - \int_X fg\right| &\leq \int_{X\setminus A} |f_n - f|\ |g| + \int_{A\setminus B} |f_n - f|\ |g| + \int_B |f_n - f|\ |g| \\
&\leq ||f_n - f||_p \left(\int_{X\setminus A} |g|^q\right)^{1/q} + ||f_n - f||_p \left(\int_{A\setminus B} |g|^q\right)^{1/q} \\
&+ \frac{\epsilon}{3\mu(B)^{1/p}||g||_q} \left(\int_B |g|^q\right)^{1/q} \left(\int_B 1^p\right)^{1/p} \\
&\leq 2C_0 \left(\int_{X\setminus A} |g|^q\right)^{1/q} + 2C_0 \left(\int_{A\setminus B} |g|^q\right)^{1/q} + \frac{\epsilon}{3} \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

b. First, a counterexample in $L^1(\mathbb{R}, m)$. Consider

$$f_n = n\ \chi_{(0,1/n)}.$$

We have $||f_n||_1 = 1$ for all positive integers n, and $f_n \to f$ a.e., where $f = 0$. However, $\lim \int f_n \cdot 1 = 1$ but $\int f \cdot 1 = 0$. Therefore f_n does not converge weakly to 0 in L^1.

Next, a counterexample in l^1. Consider

$$f_n(i) = \begin{cases} 0 & \text{if } i \neq n, \\ 1 & \text{if } i = n. \end{cases}$$

We see that $||f_n||_{l^1} = 1$ for all positive integers n, and $f_n \to f$ a.e., where $f = 0$. But $\lim \int f_n \cdot 1\ d\mu = 1$ and $\int f \cdot 1\ d\mu = 0$, where μ is the counting measure. Therefore f_n does not converge to 0 weakly in l^1.

Now we show that if $\sup_n ||f_n||_\infty = C_0 < \infty$, $f_n \to f$ a.e., and μ is $\sigma-$finite, then $f_n \to f$ in the weak* sense on L^∞. The first thing we show is that $||f||_\infty < \infty$. Indeed, for a.e. x, there exists a positive integer n such that $|f_n(x) - f(x)| < 1$. Then

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq 1 + C_0.$$

Hence $||f||_\infty \leq 1 + C_0$.

By the $\sigma-$finite condition, $L^\infty = (L^1)^*$. Let $g \in L^1$ and $\epsilon > 0$.

By fact (i), choose $\delta > 0$ such that

$$\int_E |g| < \frac{\epsilon}{3(2C_0 + 1)},$$

when $\mu(E) < \delta$. By fact (ii), choose $A \subset X$ such that $\mu(A) < \infty$ and

$$\int_{X \setminus A} |g| < \frac{\epsilon}{3(2C_0 + 1)}.$$

By fact (iii), choose $B \subset A$ and $N \in \mathbb{N}$ such that $\mu(A \setminus B) < \delta$ and

$$|f_n(x) - f(x)| < \frac{\epsilon}{3||g||_1},$$

for all $x \in B$ and $n \geq N$. With this setup, when $n \geq N$ we have

$$\begin{aligned}\left|\int_X f_n g - \int_X fg\right| &\leq \int_{X\setminus A} |f_n - f|\ |g| + \int_{A\setminus B} |f_n - f|\ |g| + \int_B |f_n - f|\ |g| \\ &\leq ||f_n - f||_\infty \int_{X\setminus A} |g| + ||f_n - f||_\infty \int_{A\setminus B} |g| + \frac{\epsilon}{3||g||_1} \int_B |g| \\ &< (2C_0+1)\frac{\epsilon}{3(2C_0+1)} + (2C_0+1)\frac{\epsilon}{3(2C_0+1)} + \frac{\epsilon}{3||g||_1}||g||_1 = \epsilon.\end{aligned}$$

Problem 6.22
Let $X = [0, 1]$, with Lebesgue measure.

a. *Let $f_n(x) = \cos 2\pi nx$. Then $f_n \to 0$ weakly in L^2 (see Exercise 63 in §5.5), but $f_n \nrightarrow 0$ a.e. or in measure.*

b. *Let $f_n(x) = n\chi_{(0,1/n)}$. Then $f_n \to 0$ a.e. and in measure, but $f_n \nrightarrow 0$ weakly in L^p for any p.*

Solution:

a. Let $u_n(x) = \sqrt{2}\cos 2\pi nx$. We note the following two identities, where $n \neq m$.

$$\int_0^1 u_n^2 = 2\int_0^1 (\cos 2\pi nx)^2 dx = 2\int_0^1 (\frac{1}{2} + \frac{1}{2}\cos 4\pi nx)dx = 1.$$

$$\int_0^1 u_n u_m = 2\int_0^1 (\cos 2\pi nx)(\cos 2\pi mx)dx = \int_0^1 (\cos(2\pi x(n-m)) + \cos(2\pi x(n+m)))dx = 0.$$

Therefore $\{u_n\}$ is orthonormal in the Hilbert space L^2. By Exercise 63(a), $\{u_n\}$ converges weakly to 0 in L^2. Therefore, $\{(1/\sqrt{2})u_n\} = \{f_n\}$ converges weakly to 0 in L^2.

We show that $f_n \nrightarrow 0$ a.e. by contradiction. If $f_n \to 0$ a.e., then $(f_n \cdot f_n) \to 0$ a.e. Since $f_n^2 \leq 1$ is dominated by an integrable function on $[0,1]$, by the Dominated Convergence Theorem we have

$$\lim \int (\cos 2\pi nx)^2 dx = \int \lim (\cos 2\pi nx)^2 dx = 0.$$

This is clearly nonsense since $\int_0^1 \cos(2\pi nx) = 1/2$.

Finally, we show $f_n \nrightarrow 0$ in measure. We explicitely calculate the measure of the set of all $x \in [0,1]$ such that $|\cos 2\pi nx| \geq 1/2$. Denote

$$E_k = \big(2\pi k, \frac{\pi}{4} + 2\pi k\big) \cup \big(\frac{3\pi}{4} + 2\pi k, \pi + 2\pi k\big) \cup \big(\pi + 2\pi k, \frac{5\pi}{4} + 2\pi k\big) \cup \big(\frac{7\pi}{4} + 2\pi k, 2\pi(k+1)\big).$$

Then $|\cos 2\pi nx| \geq 1/2$ iff $2\pi nx \in E_k$ for some $k \in \mathbb{N}$. Therefore, using the 2π periodicity of cosine, we see that $|\cos 2\pi nx| \geq 1/2$ when $x \in [0,1]$ lies in one of the $4n$ disjoint intervals of length $1/8n$. Hence

$$\{x \in [0,1] : |\cos 2\pi nx| \geq 1/2\} = \frac{1}{2}.$$

Since the measure is constant at $1/2$ for all n, $f_n \nrightarrow 0$ in measure.

(b) First we show that $f_n(x) = n\chi_{(0,1/n)} \to 0$ a.e. Fix $x \in (0,1]$. Then $f_n(x) = 0$ for all $n > 1/x$.

Next, we show that $f_n(x) = n\chi_{(0,1/n)} \to 0$ in measure. For all $\epsilon > 0$, we have

$$\mu\{x \in [0,1] : |f_n(x)| \geq \epsilon\} \leq \mu(0,1/n) = 1/n \to 0$$

as $n \to \infty$. Lastly, we show $f_n \nrightarrow 0$ weakly in L^p for any p. Since $1 \in L^q([0,1])$, let ϕ be the bounded linear functional on L^p defined by $\phi(f) = \int_0^1 f$. Then

$$\lim \phi(f_n) = \lim \int_0^1 f_n(x)dx = \lim n\frac{1}{n} = 1.$$

Therefore, f_n does not converge weakly to zero in L^p.

Problem 6.27 (Hilbert's Inequality)
The operator $Tf(x) = \int_0^\infty (x+y)^{-1} f(y)dy$ satisfies $||Tf||_p \leq C_p ||f||_p$ for $1 < p < \infty$, where $C_p = \int_0^\infty x^{-1/p}(x+1)^{-1}dx$. (For those who know about contour integrals: Show that $C_p = \pi \csc(\pi/p)$.)

Solution:

Define a function K on $(0,\infty)\times(0,\infty)$ by $K(x,y)=(x+y)^{-1}$. Then K is a Lebesgue measurable function such that $K(\lambda x,\lambda y)=\lambda^{-1}K(x,y)$ for all $\lambda>0$. In order to apply Theorem 6.20, we must verify that $\int_0^\infty |K(x,1)|x^{-1/p}dx$ is finite. This follows from $p>1$, and the facts that $\int_0^1 x^{-\alpha}<\infty$ when $0<\alpha<1$ and $\int_1^\infty x^{-\beta}<\infty$ when $1<\beta$.

$$\int_0^\infty |K(x,1)|x^{-1/p}dx=\int_0^1 \frac{1}{x^{1/p}(x+1)}+\int_1^\infty \frac{1}{x^{1/p}(x+1)}\leq \int_0^1 \frac{1}{x^{1/p}}+\int_1^\infty \frac{1}{x^{1/p}x}<\infty.$$

Hence the conditions of Theorem 6.20 are satisfied, and the desired inequality follows directly from the theorem.

Problem 6.29

Suppose that $1\leq p<\infty$, $r>0$, and h is a nonnegative measurable function on $(0,\infty)$. Then:

$$\int_0^\infty x^{-r-1}\left[\int_0^x h(y)dy\right]^p dx\leq \left(\frac{p}{r}\right)^p\int_0^\infty x^{p-r-1}h(x)^p dx,$$

$$\int_0^\infty x^{r-1}\left[\int_x^\infty h(y)dy\right]^p dx\leq \left(\frac{p}{r}\right)^p\int_0^\infty x^{p+r-1}h(x)^p dx.$$

(Apply Theorem 6.20 with $K(x,y)=x^{\beta-1}y^{-\beta}\chi_{(0,\infty)}(y-x)$, $f(x)=x^\gamma h(x)$, and $g(x)=x^\delta h(x)$ for suitable β, γ, δ.)

Solution:

We start by showing the first inequality. Consider $K(x,y)=x^{\beta-1}y^{-\beta}\chi_{(0,\infty)}(y-x)$ and $f(x)=x^\gamma h(x)$, where $\beta=(r+1)/p$ and $\gamma=1-(r+1)/p$. If $f\notin L^p(0,\infty)$, the right-hand side of the inequality is infinite and the inequality holds trivially. Hence we assume that $f\in L^p(0,\infty)$. It is clear that K is a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that $K(\lambda x,\lambda y)=\lambda^{-1}K(x,y)$ for all $\lambda>0$. Using the notation from Theorem 6.20, we evaluate C:

$$C=\int_0^\infty |K(x,1)|x^{-1/p}dx=\int_0^\infty x^{\beta-1}\chi_{(0,\infty)}(1-x)x^{-1/p}dx=\int_0^1 x^{-1+(r/p)}dx=\frac{p}{r}.$$

By Theorem 6.20, $||Tf||_p\leq C||f||_p$, where

$$Tf=\int_0^\infty x^{\beta-1}y^{-\beta}\chi_{(0,\infty)}(y-x)x^\gamma h(x)dx=y^{-(r+1)/p}\int_0^y h(x)dx.$$

Then we have the following inequalities:

$$||\ y^{-(r+1)/p}\int_0^y h(x)dx\ ||_p^p\leq C^p\ ||\ x^{1-(r+1)/p}h(x)\ ||_p^p.$$

$$\int_0^\infty y^{-(r+1)}\left[\int_0^y h(x)dx\right]^p dy\leq \left(\frac{p}{r}\right)^p\int_0^\infty x^{p-(r+1)}h(x)^p dx.$$

To show the second inequality, we reapply the theorem with different functions $\tilde{K}$ and $\tilde{f}$. Consider $\tilde{K}(x,y) = x^{\alpha-1}y^{-\alpha}\chi_{(0,\infty)}(x-y)$ and $\tilde{f}(x) = x^{\delta}h(x)$, where $\alpha = (1-r)/p$ and $\delta = 1+(r-1)/p$. If $\tilde{f} \notin L^p(0,\infty)$, the right-hand side of the inequality is infinite and the inequality holds trivially. Hence we assume that $\tilde{f} \in L^p(0,\infty)$.

We evaluate C:

$$C = \int_0^\infty |\tilde{K}(x,1)|x^{-1/p}dx = \int_0^\infty x^{\alpha-1}\chi_{(0,\infty)}(x-1)x^{-1/p}dx = \int_1^\infty x^{-1-(r/p)}dx = \frac{p}{r}.$$

By Theorem 6.20, $||T\tilde{f}||_p \leq C||\tilde{f}||_p$, where

$$T\tilde{f} = \int_0^\infty x^{\alpha-1}y^{-\alpha}\chi_{(0,\infty)}(x-y)x^{\delta}h(x)dx = y^{(-1+r)/p}\int_y^\infty h(x)dx.$$

Then we have the following inequalities:

$$||\ y^{(-1+r)/p}\int_y^\infty h(x)dx\ ||_p^p \leq C^p\ ||\ x^{1+(r-1)/p}h(x)\ ||_p^p.$$

$$\int_0^\infty y^{r-1}\left[\int_y^\infty h(x)dx\right]^p dy \leq \left(\frac{p}{r}\right)^p \int_0^\infty x^{p+r-1}h(x)^p dx.$$

Problem 6.31 (A Generalized Holder's Inequality)
Suppose that $1 \leq p_j \leq \infty$ *and* $\sum_1^n p_j^{-1} = r^{-1} = 1$. *If* $f_j \in L^{p_j}$ *for* $j = 1,\ldots,n$, *then* $\prod_1^n f_j \in L^r$ *and* $||\prod_1^n f_j||_r \leq \prod_1^n ||f_j||_{p_j}$. *(First do the case n=2.)*

Solution:
We do the case $n=2$ and then proceed by induction. When $n=2$, we apply Holder's inequality with conjugate exponents p_1/r and p_2/r.

$$||f_1f_2||_r^r = \int |f_1f_2|^r \leq ||\ |f_1|^r||_{p_1/r}\ ||\ |f_2|^r||_{p_2/r} = ||f_1||_{p_1}^r||f_2||_{p_2}^r.$$

The result follows by taking rth roots of both sides.

Suppose the result is true for $(n-1)$. First we Holder's inequality with conjugate exponents p_n/r and $(1/r)(\sum_1^{n-1} p_j^{-1})^{-1}$.

$$\begin{aligned} ||\prod_1^n f_j||_r^r &= \int |f_n|^r \prod_{j=1}^{n-1}|f_j|^r \\ &\leq ||\ |f_n|^r||_{p_n/r}\ ||\prod_{j=1}^{n-1}|f_j|^r||_{(1/r)(\sum_1^{n-1} p_j^{-1})^{-1}} \\ &= ||f_n||_{p_n}^r\ ||\prod_{j=1}^{n-1} f_j||^r_{(\sum_1^{n-1} p_j^{-1})^{-1}}. \end{aligned}$$

We can now take rth roots of both sides, and apply the induction hypothesis with r' such that $(r')^{-1} = \sum_1^{n-1} p_j^{-1}$.

$$||\prod_1^n f_j||_r \leq ||f_n||_{p_n} \; ||\prod_{j=1}^{n-1} f_j||_{r'} \leq \prod_{j=1}^{n} ||f_j||_{p_j}.$$

Problem 6.38
$f \in L^p$ iff $\sum_{-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty$.

Solution:
Suppose $f \in L^p$. Using that α_f is decreasing, and Theorem 2.15 to exchange an infinite sum and an integral, we compute the following:

$$\begin{aligned}
\sum_{-\infty}^{\infty} \lambda_f(2^k)2^{kp} &= \sum_{-\infty}^{\infty} \lambda_f(2^k)\frac{p}{1-2^{-p}}\int_{2^{k-1}}^{2^k} \alpha^{p-1}d\alpha \\
&\leq \sum_{-\infty}^{\infty} \frac{p}{1-2^{-p}}\int_{2^{k-1}}^{2^k} \lambda_f(\alpha)\alpha^{p-1}d\alpha \\
&= \frac{p}{1-2^{-p}}\int_0^{\infty} \lambda_f(\alpha)\alpha^{p-1}d\alpha \\
&= \frac{1}{1-2^{-p}}\int |f|^p < \infty
\end{aligned}$$

where the last step follows from Proposition 6.24. On the other hand, if $\sum_{-\infty}^{\infty} 2^{kp}\lambda_f(2^k) < \infty$, we have the following:

$$\begin{aligned}
\int |f|^p &= p\int_0^{\infty} \lambda_f(\alpha)\alpha^{p-1}d\alpha \\
&= p\sum_{-\infty}^{\infty}\int_{2^k}^{2^{k+1}} \lambda_f(\alpha)\alpha^{p-1}d\alpha \\
&\leq p\sum_{-\infty}^{\infty}\lambda_f(2^k)\int_{2^k}^{2^{k+1}} \alpha^{p-1}d\alpha \\
&= (2^p-1)\sum_{-\infty}^{\infty}\lambda_f(2^k)2^{kp} < \infty
\end{aligned}$$

Problem 6.39
If $f \in L^p$, then $\lim_{\alpha\to 0}\alpha^p\lambda_f(\alpha) = \lim_{\alpha\to\infty}\alpha^p\lambda_f(\alpha) = 0$. (First suppose f is simple.)

Solution:
First, suppose $f = \sum_1^n a_j\chi_{E_j}$, where the E_j are disjoint and $\mu(E_j) < \infty$ for all j. Let $a_{\max} = \max\{a_j\}$ and $a_{\min} = \min\{a_j\}$. Then when $\alpha < a_{\min}$, $\lambda_f(\alpha) = \sum \mu(E_j) < \infty$. Hence

$$\lim_{\alpha\to 0} \alpha^p \lambda_f(\alpha) = \lim_{\alpha\to 0} \ \alpha^p\big(\sum \mu(E_j)\big) = 0.$$

On the other hand, when $\alpha > a_{\max}$, then $\lambda_f(\alpha) = 0$. Therefore, $\lim_{\alpha\to\infty} \alpha^p\lambda_f(\alpha) = 0$.

We now take an arbitrary $f \in L^p$. We show that there exists a simple function $\phi = \sum_1^n a_j \chi_{E_j}$, where E_j are disjoint and $\mu(E_j) < \infty$ for all j, such that $||f-\phi||_p^p < \epsilon$ for every $\epsilon > 0$. By Theorem 2.10, there is a sequence $\{\phi_n\}$ of simple functions such that $\phi_n \to f$ a.e. and $|\phi_n| \le |f|$. Then every $\phi_n \in L^p$, and assuming $\phi_n = \sum_1^n a_j\chi_{E_j}$ where the E_j are disjoint, we see $\sum_1^n |a_j|^p \mu(E_j) = \int |\phi_n|^p \le ||f||_p^p < \infty$. Hence $\mu(E_j) < \infty$ for all j. Since

$$|f - \phi_n|^p \le 2^p(|f|^p + |\phi_n|^p) \le 2^{p+1}|f|^p,$$

we have that $|f-\phi_n|^p$ is dominated by an L^1 function, hence by dominated convergence,

$$\lim \int |f-\phi_n|^p = \int \lim |f-\phi_n|^p = 0.$$

Therefore, we can choose $\phi = \sum_1^n a_j\chi_{E_j}$, where $\mu(E_j) < \infty$ for all j, such that $||f-\phi||_p^p < \epsilon/2^{p+1}$. By the above result, for all α small enough, we have $\alpha^p\lambda_\phi(\alpha) < \epsilon/2^{p+1}$. Now, we can write $f = (f-\phi)+\phi$ and apply Proposition 6.22(d) and Chebyshev's Inequality:

$$\begin{aligned}
\limsup_{\alpha\to 0^+} \alpha^p\lambda_f(\alpha) &\le \limsup_{\alpha\to 0^+} \ \{\ \alpha^p\lambda_{f-\phi}(\tfrac{1}{2}\alpha) + \alpha^p\lambda_\phi(\tfrac{1}{2}\alpha)\ \} \\
&\le \limsup_{\alpha\to 0^+} \ \{\ 2^p||f-\phi||_p^p + 2^p\alpha^p\lambda_\phi(\alpha)\ \} \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Therefore, $\lim_{\alpha\to 0}\alpha^p\lambda_f(\alpha) = 0$. Similarly, for all α large enough, we have $\alpha^p\lambda_\phi(\alpha) < \epsilon/2^{p+1}$. The same calculation shows:

$$\begin{aligned}
\limsup_{\alpha\to \infty} \alpha^p\lambda_f(\alpha) &\le \limsup_{\alpha\to \infty} \ \{\ \alpha^p\lambda_{f-\phi}(\tfrac{1}{2}\alpha) + \alpha^p\lambda_\phi(\tfrac{1}{2}\alpha)\ \} \\
&\le \limsup_{\alpha\to \infty} \ \{\ 2^p||f-\phi||_p^p + 2^p\alpha^p\lambda_\phi(\alpha)\ \} \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Additional Problem from Class

Let $f : X \to \mathbb{C}$ be measurable, and let $E = \{p : \int_X |f|^p = ||f||_p^p < \infty\}$. It follows from a result in Folland that the set E is connected.

i) *Prove that the function $\phi(p) = \log(||f||_p^p)$ is convex in the interior of E, and that ϕ is continuous on E.*

ii) *Can E consist of a single point? Can it be any connected subset of $(0,\infty)$?*

iii) *Let* $||f||_r < \infty$ *for some* $r < \infty$. *Prove that* $\lim_{r\to\infty} ||f||_r = ||f||_\infty$.

Solution:
(i) Let $t \in (0,1)$, and $p, q \in E$. By Holder's Inequality, we know that

$$\int |f|^{pt}|f|^{q(1-t)} \leq ||\ |f|^{pt}\ ||_{1/t}\ ||\ |f|^{q(1-t)}\ ||_{1/(1-t)}.$$

Therefore, we can take the logarithm and obtain

$$\begin{aligned}\phi(tp+(1-t)q) &= \log\big(\int |f|^{tp+(1-t)q}\big)\\ &\leq \log\Big(\big(\int |f|^p\big)^t\big(\int |f|^q\big)^{1-t}\Big)\\ &= t\log\big(\int |f|^p\big) + (1-t)\log\big(\int |f|^q\big)\\ &= t\phi(p) + (1-t)\phi(q).\end{aligned}$$

Next, we show continuity of ϕ by showing sequential continuity. Suppose $\{q_n\}$ is a sequence in E converging to a point $p \in E$. If p is in the interior of E, there exists a $\delta > 0$ such that $(p-\delta, p+\delta) \subset E$ and $q_n \in (p-\delta, p+\delta)$ for all $n > N$ for some $N \in \mathbb{N}$. Then we have for all $n > N$:

$$|f|^{q_n} = |f|^{q_n}\chi_{|f|\geq 1} + |f|^{q_n}\chi_{|f|\leq 1} \leq |f|^{p+\delta}\chi_{|f|\geq 1} + |f|^{p-\delta}\chi_{|f|\leq 1} \leq |f|^{p+\delta} + |f|^{p-\delta}.$$

By the Dominated Convergence Theorem, $\lim \int |f|^{q_n} = \int |f|^p$. Therefore, by continuity of the logarithm, $\lim \phi(q_n) = \phi(p)$ and ϕ is continuous in the interior of E.

The case when p is at an endpoint of E is similar. First we do the case when $q_n \leq p$ (ie p is a right endpoint of E). Then there exists a $\delta > 0$ such that $(p-\delta, p] \subset E$ and $q_n \in (p-\delta, p]$ for all $n > N$ for some $N \in \mathbb{N}$. Then we have for all $n > N$:

$$|f|^{q_n} = |f|^{q_n}\chi_{|f|\geq 1} + |f|^{q_n}\chi_{|f|\leq 1} \leq |f|^{p}\chi_{|f|\geq 1} + |f|^{p-\delta}\chi_{|f|\leq 1} \leq |f|^{p} + |f|^{p-\delta}.$$

By the Dominated Convergence Theorem, $\lim \phi(q_n) = \phi(p)$.

When p is a left endpoint of E, then $q_n \geq p$, and there exists a $\delta > 0$ such that $[p, p+\delta) \subset E$ and $q_n \in [p, p+\delta)$ for all $n > N$ for some $N \in \mathbb{N}$. Then we have for all $n > N$:

$$|f|^{q_n} = |f|^{q_n}\chi_{|f|\geq 1} + |f|^{q_n}\chi_{|f|\leq 1} \leq |f|^{p+\delta}\chi_{|f|\geq 1} + |f|^{p}\chi_{|f|\leq 1} \leq |f|^{p+\delta} + |f|^{p}.$$

By the Dominated Convergence Theorem, $\lim \phi(q_n) = \phi(p)$.

(ii) We construct a function $f : (0,\infty) \to \mathbb{C}$ such that $E = \{1\}$. We first use the fact that for $\alpha > 0$, we have

$$\int_0^\infty \frac{1}{x^\alpha}\chi_{(0,1)} < \infty$$

if and only if $\alpha < 1$. We then consider the function

$$f = \sum_{n=1}^\infty \frac{a_n}{2^n} \frac{1}{x^{(1+1/n)^{-1}}} \chi_{(0,1)},$$

where

$$a_n = \Big|\Big|\frac{1}{x^{(1+1/n)^{-1}}} \chi_{(0,1)}\Big|\Big|_1.$$

Then f is a series of functions in L^1 that converges absolutely, so by completeness, $f \in L^1$. However, $f \notin L^p$ for any $p > 1$.

Next, we use the fact that for $\beta > 0$, we have

$$\int_0^\infty \frac{1}{x^\beta}\chi_{(1,\infty)} < \infty$$

if and only if $\beta > 1$. We then consider the function

$$g = \sum_{n=1}^\infty \frac{b_n}{2^n} \frac{1}{x^{(1+1/n)}} \chi_{(1,\infty)},$$

where

$$b_n = \Big|\Big|\frac{1}{x^{(1+1/n)}} \chi_{(1,\infty)}\Big|\Big|_1.$$

Then g is a series of functions in L^1 that converges absolutely, so by completeness, $g \in L^1$. However, $g \notin L^p$ for any $p < 1$.

Therefore, the function $f + g$ is in L^1, but not in L^p for any $p \in (0,1) \cup (1,\infty)$.

We now show how to get any interval $E = (a,b)$, again using $X = (0,\infty)$. Let $f(x) = x^{-1/b}$ when $0 < x < 1$ and identically zero otherwise. Let $g(x) = x^{-1/a}$ when $x > 1$ and identically zero otherwise. Then $f + g$ is such that $E = (a,b)$.

(iii) First, suppose $f \in L^\infty$. Then

$$\begin{aligned}\lim_{q\to\infty} ||f||_q &= \lim_{q\to\infty} \Big(\int |f|^{q-r}|f|^r\Big)^{1/q} \\ &\le \lim_{q\to\infty} \Big(\int ||f||_\infty^{q-r}|f|^r\Big)^{1/q} \\ &= \lim_{q\to\infty} ||f||_\infty^{(q-r)/q}\Big(\int |f|^r\Big)^{1/q} \\ &= ||f||_\infty.\end{aligned}$$

Therefore, if the limit exists we have $\lim_{q\to\infty} ||f||_q \le ||f||_\infty$. Now, for all $\epsilon > 0$, there is a set $E \subset X$ of positive measure such that $|f(x)| \ge ||f||_\infty - \epsilon$ for all $x \in E$. Then when $q > r$ we have

$$||f||_q \ge \Big(\int_E |f|^q\Big)^{1/q} \ge (||f||_\infty - \epsilon)(\mu(E))^{1/q}.$$

Now $\mu(E)$ is finite, so taking the limit as $q \to \infty$ we get $\lim_{q\to\infty} ||f||_q \ge ||f||_\infty - \epsilon$. Since this holds for all $\epsilon > 0$, we conclude that $\lim_{q\to\infty} ||f||_q = ||f||_\infty$.

Next, suppose $||f||_\infty = \infty$. I have a marvelous proof of this case, which this margin is too narrow to contain.