Folland: Real Analysis, Chapter 7
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Problem 7.2
Let μ be a Radon measure on X.

a. *Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Then N is open and $\mu(N) = 0$. The complement of N is called the support of μ.*

b. *$x \in \operatorname{supp}(\mu)$ iff $\int f d\mu > 0$ for every $f \in C_c(X, [0,1])$ such that $f(x) > 0$.*

Solution:
(a) Since N is the union of open sets, we know that N is open. Therefore, by inner regularity we have

$$\mu(N) = \sup\{\mu(K) : K \subset N,\ K \text{ compact}\}.$$

For any $K \subset N$ such that K is compact, we see that K is covered by finitely many sets $U_i \subset X$, $i = 1, \ldots, n$, such that $\mu(U_i) = 0$. Therefore

$$\mu(K) \leq \sum_{i=1}^{n} \mu(U_i) = 0.$$

Since the measure of N is the supremum over all such $\mu(K)$, we have $\mu(N) = 0$.

(b) Suppose $x \in \operatorname{supp}(\mu)$. Take $f \in C_c(X, [0,1])$ such that $f(x) = c > 0$. Now, $V = f^{-1}((c/2, 1))$ is an open set by continuity. Since $x \in V$, we have $\mu(V) > 0$. Hence

$$\int f d\mu \geq \int_V f d\mu \geq \frac{c}{2}\mu(V) > 0.$$

On the other hand, let $x \in X$ and suppose $\int f d\mu > 0$ for every $f \in C_c(X, [0,1])$ such that $f(x) > 0$. Since singletons are compact, $\{x\}$ is compact. Take any open set U such that $x \in U$. By Urysohn's lemma, there exists a function $g \in C_c(X, [0,1])$ such that $g(x) = 1$ and $\operatorname{supp}(g) \subseteq U$. Then by formula (7.3) in the statement of the Riesz Representation theorem,

$$\mu(U) = \sup\{\int f d\mu : f \in C_c(X), f \prec U\} \geq \int g d\mu > 0.$$

Problem 7.4
Let X be a LCH space.

a. *If $f \in C_c(X, [0, \infty))$, then $f^{-1}([a, \infty))$ is a compact G_δ set for all $a > 0$.*

b. *If $K \subset X$ is a compact G_δ set, there exists $f \in C_c(X, [0,1])$ such that $K = f^{-1}(\{1\})$.*

c. *The σ−algebra $\mathcal{B}_X^0$ of Baire sets is the σ−algebra generated by the compact G_δ sets.*

Solution:
(a) Since f is continuous, $f^{-1}(a-1/n,\infty)$ is open for all positive integers n. By the equality

$$f^{-1}([a,\infty)) = \bigcap_{n=1}^{\infty} f^{-1}(a-\frac{1}{n},\infty),$$

we see that $f^{-1}([a,\infty))$ is a G_δ set.

To show compactness, we first notice that by continuity $f^{-1}([a,\infty))$ is closed. Since $a>0$, we have $f^{-1}([a,\infty)) \subseteq \mathrm{supp} f$. Since the support of f is compact, and closed subsets of compact sets are compact, we see that $f^{-1}([a,\infty))$ is compact.

(b) Write

$$K = \bigcap_{i=1}^{\infty} U_i,$$

where U_i are open sets. By Proposition 4.31, there exists a precompact open W such that $K \subset W \subset U_1$. Define the open sets V_n as

$$V_n = \left(\bigcap_{i=1}^{n} U_i\right) \cap W.$$

Notice that $K = \cap V_n$ and that $V_1 \supset \cdots \supset V_i \supset V_{i+1} \supset \ldots$. From Urysohn's lemma, for each positive integer n, there is an $f_n \in C_c(X)$ such that $0 \le f_n \le 1$, $\mathrm{supp}(f) \subseteq V_n$ and $f=1$ on K. Define

$$f = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k.$$

The function $f(x)$ is well-defined since the pointwise sum converges absolutely:

$$\sum_{k=1}^{\infty} \frac{1}{2^k}|f_k(x)| \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$$

Actually, the sum is absolutely convergent in the uniform norm:

$$\sum_{k=1}^{\infty} \frac{1}{2^k}||f_k||_u \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

By Theorem 5.1, since $BC(X)$ is a Banach space, every absolutely convergent sequence in $BC(X)$ converges, hence $f \in BC(X)$. The support of each f_n is contained in $V_n \subset W$, and hence $\mathrm{supp} f \subset \overline{W}$. Since W is precompact, f has compact support. Since $0 \le f \le 1$, we have $f \in C_c(X,[0,1])$.

We now show $K = f^{-1}(\{1\})$. Indeed, if $x \in K$, then $f_n(x) = 1$ for all positive integers n, hence $f(x) = \sum \frac{1}{2^k} = 1$. If $x \notin K$, then there exist a positive integer i such that $x \notin V_j$ for all integers $j \geq i$. Then $x \notin \text{supp} f_j \subseteq V_j$ and $f_j(x) = 0$. Therefore

$$f(x) = \sum_{k=1}^{i} \frac{1}{2^k} f_k < 1.$$

Hence $x \notin f^{-1}(\{1\})$.

(c) The $\sigma-$algebra $\mathcal{B}_X^0$ of Baire sets is by definition generated by the sets $(\text{Re} f)^{-1}([a,\infty))$, $(\text{Im} f)^{-1}([a,\infty))$, for all $a \in \mathbb{R}$ and all $f \in C_c(X)$. Let $\mathcal{A}$ denote the $\sigma-$algebra generated by compact G_δ sets.

Let K be a compact G_δ set. Then $K = f^{-1}(\{1\}) = f^{-1}([1,\infty))$ for some $f \in C_c(X,[0,1])$ by part (b). Then $K \in \mathcal{B}_X^0$, and therefore $\mathcal{A} \subset \mathcal{B}_X^0$ by Lemma 1.1.

On the other hand, let $f \in C_c(X)$. First, we assume that f is a non-negative real-valued function. In this case, if $a > 0$, from part (a) we know that $f^{-1}([a,\infty)) \in \mathcal{A}$, and $f^{-1}([-a,\infty) = f^{-1}([0,\infty)) = X \in \mathcal{A}$. Therefore $f^{-1}([a,\infty)) \in \mathcal{A}$ for all $a \in \mathbb{R}$.

Next, suppose $f \in C_c(X)$ is real-valued. We can decompose f as

$$f = f\ \chi_{f \geq 0} - |f|\ \chi_{f<0} := f^+ - f^-.$$

Notice that f^+, f^- are non-negative real-valued functions in $C_c(X)$. If $a > 0$, we have $f^{-1}([a,\infty)) = (f^+)^{-1}([a,\infty)) \in \mathcal{A}$. Next, notice $f^{-1}([0,\infty)) = (f^-)^{-1}(\{0\}) = ((f^-)^{-1}((0,\infty]))^c$, and since

$$(f^-)^{-1}((0,\infty]) = \bigcup_{n=1}^{\infty} (f^-)^{-1}([1/n,\infty)),$$

we have $f^{-1}([0,\infty)) \in \mathcal{A}$. Lastly, we have

$$f^{-1}([-a,\infty)) = (f^-)^{-1}([0,a]) = \big((f^-)^{-1}((a,\infty))\big)^c = \left(\bigcup_{n=1}^{\infty} (f^-)^{-1}([a+1/n,\infty))\right)^c.$$

Therefore, $f^{-1}([a,\infty)) \in \mathcal{A}$ for all $a \in \mathbb{R}$.

If $f \in C_c(X)$ is complex-valued, we decompose f into its real and imaginary components, which are both real-valued functions in $C_c(X)$. Then $(\text{Re} f)^{-1}([a,\infty)) \in \mathcal{A}$, and $(\text{Im} f)^{-1}([a,\infty)) \in \mathcal{A}$, for all $a \in \mathbb{R}$. It follows that $\mathcal{B}_X^0 \subset \mathcal{A}$ by Lemma 1.1.

Problem 7.10

If μ is a Radon measure and $f \in L^1(\mu)$ is real-valued, for every $\epsilon > 0$ there exists an LSC function g

and USC function h such that $h \leq f \leq g$ and $\int (g-h)d\mu < \epsilon$.

Solution:

First, we decompose f into its positive and real parts:

$$f = f\ \chi_{f \geq 0} - |f|\ \chi_{f<0} := f^+ - f^-.$$

Since $f \in L^1(\mu)$, the sets $\{x : f^+ > 0\}$ and $\{x : f^- > 0\}$ are σ-finite by Proposition 2.20. Hence by Proposition 7.14, there exists functions g_1, g_2 such that g_1 is LSC and $g_1 \geq f^+$, g_2 is USC and $g_2 \leq f^-$, and

$$\frac{\epsilon}{4} + \int f^+ d\mu \geq \int g_1 d\mu,$$
$$-\frac{\epsilon}{4} + \int f^- d\mu \leq \int g_2 d\mu.$$

We know that $-g_2$ is LSC, so by Proposition 7.11 we have that $g := g_1 - g_2$ is LSC. Futhermore, $g \geq f$.

Again by Proposition 7.14, there exists functions h_1, h_2 such that h_1 is USC and $h_1 \leq f^+$, h_2 is LSC and $h_2 \geq f^-$, and

$$-\frac{\epsilon}{4} + \int f^+ d\mu \leq \int h_1 d\mu,$$
$$\frac{\epsilon}{4} + \int f^- d\mu \geq \int h_2 d\mu.$$

Since $-h_2$ is USC, we have that $h := h_1 - h_2$ is USC. Furthermore, $f \geq h$. We now compute

$$\begin{aligned}\int (g-h)d\mu &= \int g_1 d\mu - \int g_2 d\mu - \int h_1 d\mu + \int h_2 d\mu \\ &\leq \int f^+ d\mu + \frac{\epsilon}{4} - \int f^- d\mu + \frac{\epsilon}{4} - \int f^+ d\mu + \frac{\epsilon}{4} + \int f^- d\mu + \frac{\epsilon}{4} \\ &= \epsilon.\end{aligned}$$

Problem 7.11

Suppose that μ is a Radon measure on X such that $\mu(\{x\}) = 0$ for all $x \in X$, and $A \in \mathcal{B}_X$ satisfies $0 < \mu(A) < \infty$. Then for any α such that $0 < \alpha < \mu(A)$ there is a Borel set $B \subset A$ such that $\mu(B) = \alpha$.

Solution:

We define

$$\mathcal{P} = \{B \subseteq A \ : \ B = U \cap A, \ U \text{ open, and } \mu(B) \leq \alpha\}.$$

First, we show that $\mathcal{P}$ contains an element that is not the empty set. Since A is non-empty, there exists an element $x \in A$, and

$$0 = \mu(\{x\}) = \inf\{\mu(V) : \{x\} \subseteq V, \ V \text{ open}\}.$$

Therefore, there exists an open set V containing x such that $\mu(V) < \alpha$. Hence $\emptyset \neq V \cap A \in \mathcal{P}$.

We can partially order $\mathcal{P}$ by inclusion. Let $\mathcal{C}$ be a totally ordered non-empty subset of $\mathcal{P}$. We show that $\mathcal{C}$ has an upper bound in $\mathcal{P}$. Indeed, take

$$U = \bigcup_{B_\gamma \in \mathcal{C}} B_\gamma.$$

It is clear that U is an upper bound of $\mathcal{C}$. Since the union of open sets is open, we have that $U \cap A$ is open in A. Since $\mu(U) \leq \mu(A) < \infty$, we see that U is a σ-finite set, and hence by Proposition 7.5 we have

$$\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}.$$

For any K compact such that $K \subseteq U$, we see that K is covered by finitely many $B_\gamma \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered by inclusion, $K \subseteq B_\gamma$ for some $B_\gamma \in \mathcal{C}$. Hence

$$\mu(K) \leq \mu(B_\gamma) \leq \alpha.$$

By taking the supremum over all such K, we see that $\mu(U) \leq \alpha$. Hence $U \in \mathcal{P}$. By Zorn's lemma, the set $\mathcal{P}$ contains a maximal element B.

We claim that $\mu(B) = \alpha$. By contradiction, assume that $\alpha - \mu(B) > 0$. Take $x \in A \backslash B$, which exists since the measure of $A \backslash B$ is positive. By outer regularity, there exists an open set V such that $x \in A \cap V$ and

$$\mu(V) < \alpha - \mu(B).$$

But then $B \cup (A \cap V)$ is open in A and

$$\mu(B \cup (A \cap V)) \leq \mu(B) + \alpha - \mu(B) \leq \alpha.$$

Hence $B \subsetneq B \cup (A \cap V)$ and $B \cup (A \cap V) \in \mathcal{P}$, contradicting the maximality of B.

Problem 7.20

Some examples of nonreflexivity of $C_0(X)$:

a. *If $\mu \in M(X)$, let $\Phi(\mu) = \sum_{x\in X} \mu(\{x\})$. This sum is well defined, and $\Phi \in M(X)^*$. If there exists a nonzero $\mu \in M(X)$ such that $\mu(\{x\}) = 0$ for all $x \in X$, then Φ is not in the image of $C_0(X)$ in $M(X)^* \cong C_0(X)^{**}$.*

b. *At the other extreme, let $X = \mathbb{N}$ with the discrete topology; then $C_0(X)^* \cong l^1$ and $(l^1)^* \cong l^\infty$. (Note: $C_0(\mathbb{N})$ is usually denoted by c_0.)*

Solution:

(a) First, we show that the sum is well defined. Let

$$A_n = \{x \in X : \mu(\{x\}) \geq 1/n\}.$$

If $\text{card}(A_n)$ is infinite, then letting A_n^* be a countably infinite subset of A_n we see that

$$\mu(X) \geq \sum_{x\in A_n^*} \mu(\{x\}) \geq (1/n) + (1/n) + (1/n) + \cdots = \infty.$$

Hence $\text{card}(A_n)$ is finite for all positive integers n. Now define

$$A_\mu = \{x \in X : \mu(\{x\}) > 0\}.$$

Since $A_\mu = \cup A_n$, we see that A_μ is a countable set. Hence by countable additivity we have

$$|\Phi(\mu)| \leq \sum_{x\in A_\mu} |\mu|(\{x\}) = |\mu|(A_\mu) \leq |\mu|(X) = ||\mu|| < \infty.$$

This shows that the sum is well-defined, and furthermore that Φ is a continuous operator on $M(X)$. Linearity of Φ follows from the fact that terms in an absolutely convergent series can be rearranged:

$$\Phi(c\mu + \nu) = \sum_{x\in A_{\mu+\nu}} (c\mu + \nu)(\{x\}) = \sum_{x\in A_{\mu+\nu}} (c\mu)(\{x\}) + \nu(\{x\}) = c\sum_{x\in A_\mu} \mu(\{x\}) + \sum_{x\in A_\nu} \nu(\{x\}).$$

Hence $\Phi \in M(X)^*$.

Suppose there exists a nonzero $\mu \in M(X)$ such that $\mu(\{x\}) = 0$ for all $x \in X$, and that $\Phi \in C_0(X)$. Recall the Dirac measure δ_x defined by $\delta_x(y) = 0$ if $y \neq x$ and $\delta_x(x) = 1$. The Dirac measure is a Radon measure, and by definition of Φ, we have $\Phi(\delta_x) = 1$. However, as an element of $C_0(X)$, we have the action

$$\Phi(\delta_x) = \int \Phi \; d\delta_x = \Phi(x).$$

From this we conclude that $\Phi(x) = 1$ for all $x \in X$. But by definition of Φ, we have $\Phi(|\mu|) = 0$ since $|\mu|\{x\} = 0$ for all $x \in X$. As an element of $C_0(X)$ we have the action

$$\Phi(|\mu|) = \int 1 \cdot d|\mu| = |\mu|(X).$$

This forces us to conclude that $|\mu|(X) = 0$, which implies that for all Borel sets E we have $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X) = 0$, a contradiction since μ is non-zero.

(b) We can identify functions $f \in C_0(\mathbb{N})$ with sequences $\{f_n\}$ such that $f_n \to 0$ as $n \to \infty$. The norm $||f||$ is given by $||f|| = \max_n |f_n|$. We define a map $\Psi : l^1 \to C_0(\mathbb{N})^*$ given by

$$\Psi(\{a_n\})(f) = \sum_{n=1}^{\infty} a_n f_n.$$

This is well defined, since

$$\sum_{n=1}^{\infty} |a_n| \; |f_n| \leq ||f|| \sum_{n=1}^{\infty} |a_n| < \infty.$$

We show Ψ is a bijection by producing an inverse. First, we define

$$g_i(n) = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{if } n \neq i \end{cases}$$

Then $\Psi^{-1}(\mu) = \{a_n\}$, where

$$a_i = \Psi(\{a_n\})(g_i) = \int g_i d\mu = \mu(\{i\}).$$

This is well defined, since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |\mu(\{n\})| \leq \sum_{n=1}^{\infty} |\mu|(\{n\}) \leq |\mu|(\mathbb{N}) < \infty.$$

It is easy to see that Ψ is linear, hence it only remains to show that Ψ is an isometry. We have already seen that

$$|\Psi(\{a_n\})(f)| \leq ||f|| \cdot ||\{a_n\}||_{l^1},$$

hence $||\Psi(\{a_n\})|| \leq ||\{a_n\}||$. We define the functions $h_i \in C_0(\mathbb{N})$ as (using the sgn function defined on page 46):

$$h_i(n) = \begin{cases} (\text{sgn } a_n) & \text{if } n \leq i \\ 0 & \text{if } n > i \end{cases}$$

Now since

$$||\Psi(\{a_n\})|| = \sup_{f \in C_0(\mathbb{N}), ||f||=1} |\Psi(\{a_n\})(f)|,$$

and $||h_i|| = 1$, we conclude that

$$||\Psi(\{a_n\})|| \geq |\Psi(\{a_n\})(h_i)| = \sum_{n=1}^{i} |a_n|.$$

Letting $i \to \infty$, we see that $||\Psi(\{a_n\})|| = ||\{a_n\}||$ and hence Ψ is an isometric isomorphism.

The fact that $(l^1)^* \equiv l^\infty$ is a direct consequence of Theorem 6.15.

Problem 7.24
Find examples of sequences $\{\mu_n\}$ in $M(\mathbb{R})$ such that
a. $\mu_n \to 0$ *vaguely, but* $||\mu_n|| \nrightarrow 0$.
b. $\mu_n \to 0$ *vaguely, but* $\int f d\mu_n \nrightarrow 0$ *for some bounded measurable function f with compact support.*
c. $\mu_n \geq 0$ *and* $\mu_n \to 0$ *vaguely, but there exists* $x \in \mathbb{R}$ *such that* $F_n(x) \nrightarrow 0$ *(notation as in Proposition 7.19).*

Solution:
(a) Let $\mu_n = \delta_n$, where δ_n is the Radon measure defined in the solution to the previous exercise. Then for all $f \in C_0(\mathbb{R})$, we have

$$\int f d\mu_n = f(n) \to 0$$

as $n \to \infty$. Hence $\mu_n \to 0$ vaguely. However, $||\mu_n|| = |\delta_n|(\mathbb{R}) = 1$ for all positive integers n.

(b) Let $\mu_n = \delta_{1/n} - \delta_{-1/n}$. Then for any $f \in C_0(\mathbb{R})$, we have

$$\int f d\mu_n = f(1/n) - f(-1/n) \to 0$$

as $n \to \infty$. Hence $\mu_n \to 0$ vaguely. However, the function

$$f(x) = (\text{sgn}x)\ \chi_{[-2,2]}$$

is bounded and measurable with compact support, and $\int f d\mu_n = 1 - -1 = 2$ for all positive integers n.

(c) Let $\mu_n = \delta_{-n}$. Then $\mu_n \geq 0$ and for any $f \in C_0(\mathbb{R})$, we have

$$\int f d\mu_n = f(-n) \to 0$$

as $n \to \infty$. Hence $\mu_n \to 0$ vaguely. However, $F_n(0) = \mu_n((-\infty, 0]) = 1$ for all positive integers n.

Problem 7.27

Let $C^k([0,1])$ be as in Exercise 9 in §5.1. If $I \in C^k([0,1])^*$, there exist $\mu \in M([0,1])$ and constants $c_0, \ldots, c_{k-1}$, all unique, such that

$$I(f) = \int f^{(k)} d\mu + \sum_0^{k-1} c_j f^{(j)}(0).$$

(The functionals $f \mapsto f^{(j)}(0)$ could not be replaced by any set of k functionals that separate points in the space of polynomials of degree $< k$.)

Solution:

Take any $f \in C^k([0,1])$. By Taylor's Theorem, we can Taylor expand about 0 and obtain

$$f(x) = \sum_{n=1}^{k-1} \frac{f^{(n)}(0)}{n!} x^n + \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) dt.$$

Let $I \in C^k([0,1])^*$. The bounded linear functional I is defined on a dense subset of $C[0,1]$, hence can be uniquely extended to a bounded linear functional on all of $C[0,1]$. By the Riesz Representation Theorem, there exists a Radon measure $\nu \in M([0,1])$ such that $I(f) = \int f d\nu$. Then we have

$$\begin{aligned}
I(f) &= \int_0^1 \sum_{n=1}^{k-1} \frac{f^{(n)}(0)}{n!} x^n d\nu + \int_0^1 \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\ dt\ d\nu(x) \\
&= \sum_{n=1}^{k-1} c_n f^{(n)}(0) + \int_0^1 \int_0^1 \chi_{[0,x]}(t)\ \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\ dt\ d\nu(x) \\
&= \sum_{n=1}^{k-1} c_n f^{(n)}(0) + \int_0^1 \int_0^1 \chi_{[0,x]}(t)\ \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\ d\nu(x)\ dt \\
&= \sum_{n=1}^{k-1} c_n f^{(n)}(0) + \int_0^1 \int_0^1 \chi_{[t,1]}(x)\ \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\ d\nu(x)\ dt \\
&= \sum_{n=1}^{k-1} c_n f^{(n)}(0) + \int_0^1 f^{(k)}(t) \int_t^1 \frac{(x-t)^{k-1}}{(k-1)!}\ d\nu(x)\ dt \\
&= \sum_{n=1}^{k-1} c_n f^{(n)}(0) + \int_0^1 f^{(k)}(t) g(t) dt.
\end{aligned}$$

where $c_n = \int_0^1 x^n/n!\ d\nu$ and $g(t) = \int_t^1 \frac{(x-t)^{k-1}}{(k-1)!}\ d\nu(x)$. In the preceding computation, the order of integrals was switched by Fubini's Theorem. From the discussion on page 223, we know that $d\mu = g(t)dt$, where dt is Lebesgue measure, defines a Radon measure.

We must now show uniqueness. For any integer $0 \leq n < k$, we substitute the function x^n into the formula for I to obtain

$$I(x^n) = n!\, c_n.$$

Therefore, the constants $c_0, \ldots, c_{k-1}$ are uniquely determined. If there exists another $\tilde{\mu} \in M([0,1])$ with the desired property, then since the $c_0, \ldots, c_{k-1}$ are unique, we see that

$$\int f^{(k)} d\mu = \int f^{(k)} d\tilde{\mu},$$

for all $f \in C^k([0,1])$. To show the two measures are equal, it suffices to show that $\mu([0,c]) = \tilde{\mu}([0,c])$ for all $c \in [0,1]$. If we let $f^{(k)} = 1$, we see that $\mu([0,1]) = \tilde{\mu}([0,1])$. Let $c \in [0,1)$. For all positive integers n such that $c + 1/n \leq 1$, we can define functions $f_n \in C^k([0,1])$ such that $f_n^{(k)} = 1$ on $[0,c]$, $f_n^{(k)} = 0$ on $[c+1/n, 1]$, and $f_n^{(k)}$ is linear on $[c, c+1/n]$. Pointwise, we have $f_n^{(k)} \to \chi_{[0,c]}$. Then by the Dominated Convergence Theorem, we see that

$$\mu([0,c]) = \int_0^c d\mu = \lim \int_0^1 f_n^{(k)} d\mu = \lim \int_0^1 f_n^{(k)} d\tilde{\mu} = \int_0^c d\tilde{\mu} = \tilde{\mu}([0,c]).$$

Problem 7.30

Let μ and ν be Radon measures on X and Y, not necessarily $\sigma-$finite. If f is a nonnegative LSC function on $X \times Y$, then $x \mapsto \int f_x \, d\nu$ and $y \mapsto \int f^y \, d\mu$ are Borel measurable and $\int f \, d(\mu \hat{\times} \nu) = \int \int f \, d\mu d\nu = \int \int f \, d\nu d\mu$.

Solution:

Warning: this solution seems sketchy to me. I don't fell great about it. Suggestions are welcome.

We define the functions $\Phi_f : X \to \mathbb{C}$ and $\Psi_f : Y \to \mathbb{C}$ as

$$\Phi_f(x) = \int f_x \, d\nu,$$
$$\Psi_f(y) = \int f^y \, d\mu.$$

By Lemma 7.24, if $f \in C_c(X \times Y)$, then Φ_f and Ψ_f are continuous. If f is a nonnegative LSC function on $X \times Y$, then so are f_x and f^y, hence by Corollary 7.13 we have

$$\begin{aligned}
\Phi_f(x) &= \int f_x \, d\nu \\
&= \sup \left\{ \int g \, d\nu : g \in C_c(Y),\ 0 \leq g \leq f_x \right\} \\
&= \sup \left\{ \int g_x \, d\nu : g \in C_c(X \times Y),\ 0 \leq g \leq f \right\} \\
&= \sup \left\{ \Phi_g(x) : g \in C_c(X \times Y),\ 0 \leq g \leq f \right\}
\end{aligned}$$

Since the supremum of a family of measurable functions is measurable, we see that $\Phi_f(x)$ is a measurable function. The same argument shows that $\Psi_f(x)$ is a measurable function. The equality of the supremums of the two different families in the previous argument is justified as follows. If $g \in C_c(X \times Y)$ and $0 \leq g \leq f$, it is immediate that $g_x \in C_c(Y)$ and $0 \leq g_x \leq f_x$. On the other hand, if $g \in C_c(Y)$ and $0 \leq g \leq f_x$, by Urysohn's lemma we can extend g to a function $\tilde{g} \in C_c(X \times Y)$ with $0 \leq \tilde{g} \leq f$, such that $\tilde{g}_x = g$.

Next, we notice that by the Fubini-Tonelli theorem for Radon products, if $g \in C_c(X \times Y)$, then

$$\int g \ d(\mu \hat{\times} \nu) = \int \int g \ d\mu d\nu = \int \int g \ d\nu d\mu$$

since the support of g has finite measure, hence the integrals can be taken over a set of finite measure. By applying Corollary 7.13, if f is nonnegative and LSC, then

$$\begin{aligned} \int f \ d(\mu \hat{\times} \nu) &= \sup \left\{ \int g \ d(\mu \hat{\times} \nu) : g \in C_c(X \times Y), \ 0 \leq g \leq f \right\} \\ &= \sup \left\{ \int \int g \ d\mu d\nu : g \in C_c(X \times Y), \ 0 \leq g \leq f \right\} \\ &= \int \int f \ d\mu d\nu \end{aligned}$$

The same argument shows $\int f \ d(\mu \hat{\times} \nu) = \int \int f \ d\nu d\mu$.