Folland: Real Analysis, Chapter 8
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Problem 8.3
Let $\eta(t) = e^{-1/t}$ for $t > 0$, $\eta(t) = 0$ for $t \leq 0$.
 a. *For $k \in \mathbb{N}$ and $t > 0$, $\eta^{(k)}(t) = P_k(1/t)e^{-1/t}$ where P_k is a polynomial of degree $2k$.*
 b. *$\eta^{(k)}(0)$ exists and equals zero for all $k \in \mathbb{N}$.*

Solution:
(a) It is clear that for $t > 0$ all derivatives $\eta^{(k)}(t)$ exist and can by computed by the usual rules of calculus. We proceed by induction. First, notice

$$\eta^{(1)}(t) = \frac{d}{dt} e^{-1/t} = \frac{1}{t^2} e^{-1/t} = P_1(1/t)e^{-1/t},$$

where $P_1 = x^2$ is a polynomial of degree 2.

Suppose $\eta^{(k)}(t) = P_k(1/t)e^{-1/t}$ for some k, where P_k is a polynomial of degree $2k$. Then

$$\eta^{(k+1)}(t) = \left(\frac{d}{dt} P_k(1/t) \right) e^{-1/t} + \frac{P_k(1/t)}{t^2} e^{-1/t} = \left(\frac{1}{t^2} P_k(1/t) - \frac{1}{t^2} P_k'(1/t) \right) e^{-1/t}.$$

This yields $\eta^{(k+1)}(t) = P_{k+1}(1/t)e^{-1/t}$ where

$$P_{k+1} = x^2 P_k - x^2 P_k'.$$

Since P_k is a polynomial of degree $2k$, $x^2 P_k$ has highest order coefficient of degree $2k+2$. On the other hand, P_k' is a polynomial of degree less than or equal to $2k-1$; therefore $x^2 P_k'$ has degree less than or equal to $2k+1$. It follows that $P_{k+1} = x^2 P_k - x^2 P_k'$ is a polynomial of degree $2(k+1)$. The proof follows by induction.

(b) We proceed again by induction. First, we compute

$$\lim_{t \to 0^+} \frac{\eta(t) - \eta(0)}{t} = \lim_{t \to 0^+} \frac{e^{-1/t}}{t} = \lim_{s \to \infty} s e^{-s} = 0.$$

The left-handed limit is easy since $\eta(t) \equiv 0$ when $t \leq 0$:

$$\lim_{t \to 0^-} \frac{\eta(t) - \eta(0)}{t} = 0.$$

Therefore, $\eta'(0)$ exists and is equal to zero.

Suppose $\eta^{(k)}(0)$ exists and equals zero. We compute $\eta^{(k+1)}(0)$. Since $\eta(t) \equiv 0$ for $t \leq 0$, we have $\eta^{(m)}(t) = 0$ for all $t < 0$ and all $m \in \mathbb{N}$. By induction hypothesis, $\eta^{(k)}(0) = 0$. Therefore we have the following left-handed limit:

$$\lim_{t\to 0^-} \frac{\eta^{(k)}(t) - \eta^{(k)}(0)}{t} = 0.$$

On the other hand, we can use (a) and the fact that e^{-x} decays faster than any polynomial to compute the following right-handed limit:

$$\lim_{t\to 0^+} \frac{\eta^{(k)}(t) - \eta^{(k)}(0)}{t} = \lim_{t\to 0^+} \frac{P_k(1/t)e^{-1/t}}{t} = \lim_{s\to\infty} sP_k(s)e^{-s} = 0.$$

This shows that $\eta^{(n+1)}(0) = 0$. By induction, $\eta^{(k)}(0)$ exists and equals zero for all $k \in \mathbb{N}$.

Problem 8.4

If $f \in L^\infty$ and $||\tau_y f - f||_\infty \to 0$ as $y \to 0$, then f agrees a.e. with a uniformly continuous function. (Let $A_r f$ be as in Theorem 3.18. Then $A_r f$ is uniformly continuous for $r > 0$ and uniformly Cauchy as $r \to 0$.)

Solution:

Define $A_r f$ as in Theorem 3.18:

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(z)dz.$$

The first step is to show that for $r > 0$, $A_r f$ is uniformly continuous. Choose $\delta > 0$ such that for all $c \in \mathbb{R}^n$ such that $|c| < \delta$, we have $||\tau_c f - f||_\infty < \epsilon$. Then when $|x - y| < \delta$, we have

$$\begin{aligned}
|A_r f(x) - A_r f(y)| &= \frac{1}{m(B(r,x))} \left| \int_{B_r(x)} f(z)dz - \int_{B_r(y)} f(z)dz \right| \\
&= \frac{1}{m(B(r,x))} \left| \int_{B_r(x)} f(z)dz - \int_{B_r(x)} f(z+y-x)dz \right| \\
&\le \frac{1}{m(B(r,x))} \int_{B_r(x)} |f(z) - f(z-(x-y))|dz \\
&\le \frac{||\tau_{x-y} f - f||_\infty}{m(B(r,x))} \int_{B_r(x)} dz \\
&< \epsilon.
\end{aligned}$$

Next, we show that $A_r f$ is uniformly Cauchy. Choose $\delta > 0$ such that $||\tau_c f - f||_\infty < \epsilon/2$ when $|c| < \delta$. Then for all $r, s > 0$ such that $r < \delta$, $s < \delta$, for any x we have

$$\begin{aligned}
|A_r f(x) - A_s f(x)| &\leq |A_r f(x) - f(x)| + |A_s f(x) - f(x)| \\
&= \frac{1}{m(B(r,x))}\left|\int_{B_r(x)} (f(z) - f(x))dz\right| + \frac{1}{m(B(s,x))}\left|\int_{B_s(x)} (f(z) - f(x))dz\right| \\
&\leq \frac{1}{m(B(r,x))}\int_{B_r(x)} |\tau_{z-x} f(z) - f(z)|dz + \frac{1}{m(B(s,x))}\int_{B_s(x)} |\tau_{z-x} f(z) - f(z)|dz \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Taking the supremum over all points x, we see that $A_r f$ is uniformly Cauchy.

At each point x, $\{A_r f(x)\}$ is Cauchy, so we can define $\tilde{f}(x) = \lim_{r\to 0} A_r f(x)$. We show $A_r f$ converges to $\tilde{f}$ as $r \to 0$ in the uniform norm. Select $\delta > 0$ such that $||A_{s_1} f - A_{s_2} f||_u < \epsilon$ for all $0 < s_1 < \delta$, $0 < s_2 < \delta$. Then if $0 < r < \delta$, for each point x we have

$$|A_r f(x) - f(x)| = \lim_{s\to 0} |A_r f(x) - A_s f(x)| \leq \lim_{s\to 0} ||A_r f - A_s f||_u < \epsilon.$$

We now show that $\tilde{f}$ is uniformly continuous. Select $r > 0$ such that $||A_r f - \tilde{f}||_u < \epsilon/3$ and $\delta > 0$ such that $|A_r f(x) - A_r f(y)| < \epsilon/3$ when $|x - y| < \delta$. Then for any points such that $|x - y| < \delta$,

$$\begin{aligned}
|\tilde{f}(x) - \tilde{f}(y)| &\leq |\tilde{f}(x) - A_r f(x)| + |A_r f(x) - A_r f(y)| + |A_r f(y) - \tilde{f}(y)| \\
&\leq ||\tilde{f} - A_r f||_u + |A_r f(x) - A_r f(y)| + ||A_r f - \tilde{f}||_u \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

By Theorem 3.18, $A_r f \to f$ as $r \to 0$ pointwise a.e. By unicity of the limit, we must have $\tilde{f} = f$ a.e.

Problem 8.8

Suppose that $f \in L^P(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that

$$\lim_{y\to 0} ||y^{-1}(\tau_{-y} f - f) - h||_p = 0,$$

we call h the (strong) L^p derivative of f. If $f \in L^p(\mathbb{R}^n)$, L^p partial derivatives of f are defined similarly. Suppose that p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and the L^p derivative $\partial_j f$ exists. Then $\partial_j(f * g)$ exists (in the ordinary sense) and equals $(\partial_j f) * g$.

Solution:

First, we denote $\tilde{g}(x) = g(-x)$. After a change of variables, we see that $||g||_q = ||\tilde{g}||_q$. Denote the L^p derivative of f as $h \in L^p$.

Then at any point $x \in \mathbb{R}^n$, we have

$$
\begin{aligned}
|\partial_j(f*g)(x)-(h*g)(x)| &= \lim_{t\to 0}\left|\int \frac{f(x+te_j-y)g(y)-f(x-y)g(y)}{t}dy - \int h(x-y)g(y)dy\right| \\
&\le \lim_{t\to 0}\int |g(y)|\,|t^{-1}\left(f(x+te_j-y)-f(x-y)\right)-h(x-y)|dy \\
&= \lim_{t\to 0}\int |g(x-z)|\,|t^{-1}\left(f(z+te_j)-f(z)\right)-h(z)|dz \\
&\le \lim_{t\to 0}\,||\tau_{-x}\tilde{g}||_q\,||t^{-1}\left(\tau_{-te_j}f-f\right)-h||_p \\
&= ||g||_q\,\lim_{t\to 0}\,||t^{-1}\left(\tau_{-te_j}f-f\right)-h||_p = 0.
\end{aligned}
$$

Problem 8.14 (Wirtinger's Inequality)
If $f\in C^1([a,b])$ and $f(a)=f(b)=0$, then

$$\int_a^b |f(x)|^2dx \le \left(\frac{b-a}{\pi}\right)^2\int_a^b |f'(x)|^2dx.$$

(By a change of variable it suffices to assume $a=0$, $b=1/2$. Extend f to $[-1/2,1/2]$ by setting $f(-x)=-f(x)$, and then extend f to be periodic on $\mathbb{R}$. Check that f, thus extended, is in $C^1(\mathbb{T})$ and apply Parseval identity.)

Solution:

The first step is to show that, without loss of generality, we can assume that $a=0$, $b=1/2$. Suppose the inequality holds for this specific case. Then via the change of variables $x=2(b-a)z+a$, we obtain

$$
\begin{aligned}
\int_a^b |f(x)|^2dx &= 2(b-a)\int_0^{1/2}|f(2(b-a)z+a)|^2dz \\
&\le 2(b-a)\left(\frac{1}{2\pi}\right)^2\int_0^{1/2}|\frac{d}{dz}f(2(b-a)z+a)|^2dz \\
&= (2(b-a))^3\left(\frac{1}{2\pi}\right)^2\int_0^{1/2}|f'(2(b-a)z+a)|^2dz \\
&= \left(\frac{b-a}{\pi}\right)^2\int_a^b|f'(x)|^2dx.
\end{aligned}
$$

Next, we extend f to $[-1/2,1/2]$ by setting $f(-x)=-f(x)$, and then extend f to be periodic on $\mathbb{R}$ in the obvious way. To check that $f\in C^1(\mathbb{T})$, it suffices to verify that f is differentiable and its derivative is continuous at 0; all other points of the form $n/2$, $n\in\mathbb{Z}$ are done similarly. By the definition of $C^1([0,1/2])$ given in Exercise 5.9, f has a one-sided derivative at the endpoint 0:

$$\lim_{t\to 0^+}\frac{f(t)-f(0)}{t}=c.$$

The left-handed derivative can be computed using the symmetry of f:

$$\lim_{t\to 0^-} \frac{f(t)-f(0)}{t} = \lim_{t\to 0^+} \frac{f(-t)}{-t} = \lim_{t\to 0^+} \frac{-(f(t)-f(0))}{-t} = c.$$

Hence $f'(0)$ exists and is equal to c. Next, we show that $f'(x) = f'(-x)$. Indeed, $d/dx(f(-x)) = -d/dx(f(x)) = -f'(x)$, and on the other hand by the chain rule,

$$\frac{d}{dx}(f(-x)) = f'(-x)\cdot(-1).$$

Cancelling (-1) from both sides yields the result. We now show that the derivative is continuous at zero. Since $f \in C^1([0,1/2])$,

$$\lim_{t\to 0^+} f'(t) = c.$$

On the other hand,

$$\lim_{t\to 0^-} f'(t) = \lim_{t\to 0^+} f'(-t) = \lim_{t\to 0^+} f'(t) = c.$$

This completes the proof that $f \in C^1(\mathbb{T})$.

Now that we have reduced the problem to an easier case, we prove Wirtinger's inequality. Using Parseval's identity and integration by parts, we compute

$$\begin{aligned}
\int_{-1/2}^{1/2} |f(x)|^2 dx = ||f||_{L^2}^2 &= \sum_{k\in\mathbb{Z}} \Big| \int_{-1/2}^{1/2} f(x)\ e^{-i(2\pi kx)} dx \Big|^2 \\
&= \Big| \int_{-1/2}^{1/2} f(x) dx \Big| + \sum_{k\neq 0, k\in\mathbb{Z}} \Big| \int_{-1/2}^{1/2} f(x)\ e^{-i(2\pi kx)} dx \Big|^2 \\
&= \sum_{k\neq 0, k\in\mathbb{Z}} \Big| \int_{-1/2}^{1/2} f(x)\ \frac{-1}{i(2\pi k)}\ d\ (e^{-i(2\pi kx)}) \Big|^2 \\
&= \sum_{k\neq 0, k\in\mathbb{Z}} \Big| \int_{-1/2}^{1/2} \frac{f'(x)}{i(2\pi k)}\ e^{-i(2\pi kx)} dx \Big|^2 \\
&= \sum_{k\neq 0, k\in\mathbb{Z}} \frac{1}{4\pi^2 k^2} \Big| \int_{-1/2}^{1/2} f'(x)\ e^{-i(2\pi kx)} dx \Big|^2 \\
&\leq \frac{1}{4\pi^2} \sum_{k\neq 0, k\in\mathbb{Z}} \Big| \int_{-1/2}^{1/2} f'(x)\ e^{-i(2\pi kx)} dx \Big|^2.
\end{aligned}$$

Since $f = 0$ at the endpoints, we see that $\int_{-1/2}^{1/2} f' = 0$. Hence by Parseval's identity we have

$$\int_{-1/2}^{1/2} |f(x)|^2 dx \leq \frac{1}{4\pi^2} \sum_{k\in\mathbb{Z}} \Big| \int_{-1/2}^{1/2} f'(x)\ e^{-i(2\pi kx)} dx \Big|^2 = \frac{1}{4\pi^2} \int_{-1/2}^{1/2} |f'(x)|^2 dx.$$

Wirtinger's inequality then follows by symmetry:

$$\int_0^{1/2} |f|^2 = \frac{1}{2}\int_{-1/2}^{1/2} |f|^2 \le \frac{1}{2}\,\left(\frac{1/2-0}{\pi}\right)^2 \int_{-1/2}^{1/2} |f'|^2 = \left(\frac{1/2-0}{\pi}\right)^2 \int_0^{1/2} |f'|^2.$$

Problem 8.16

Let $f_k = \chi_{[-1,1]} * \chi_{[-k,k]}$.

a. *Compute* $f_k(x)$ *explicitly and show that* $||f||_u = 2$.

b. $f_k^\vee(x) = (\pi x)^{-2}\sin 2\pi kx \sin 2\pi x$, *and* $||f_k^\vee||_1 \to \infty$ *as* $k \to \infty$. *(Use Exercise 15a, and substitute* $y = 2\pi kx$ *in the integral defining* $||f_k^\vee||_1$*.)*

c. $\mathcal{F}(L^1)$ *is a proper subset of* C_0. *(Consider* $g_k = f_k^\vee$ *and use the open mapping theorem.)*

Solution:

(a) By definition of convolution, we have

$$f_k(x) = \int_{-k}^{k} \chi_{[-1,1]}(x-y)dy.$$

For $k > 0$, $k \in \mathbb{N}$, unravelling the definitions yields

$$f_k(x) = \begin{cases} 2 & \text{if } -(k-1) \le x \le k-1, \\ 0 & \text{if } x \le -(k+1), \text{ or } x \ge k+1, \\ (1+k) - x & \text{if } k-1 \le x \le k+1, \\ (1+k) + x & \text{if } -(k+1) \le x \le -(k-1). \end{cases}$$

We see that $f_k \in C_0$, and $||f_k||_u = 2$.

(b) We use Exercise 15a to compute

$$(f_k)^\vee(x) = (\chi_{[-1,1]} * \chi_{[-k,k]})^\vee = (\chi_{[-1,1]})^\vee\ (\chi_{[-k,k]})^\vee = \frac{1}{(\pi x)^2}\sin 2\pi kx \sin 2\pi x.$$

By substituting $y = 2\pi kx$, we then obtain

$$||f_k^\vee||_1 = \int_{-\infty}^{\infty} \left|\frac{1}{(\pi x)^2}\sin 2\pi kx \sin 2\pi x\right| dx = \frac{4}{\pi}\int_0^{\infty} \left|\frac{\sin y}{y}\right|\ \left|\frac{\sin(y/k)}{y/k}\right| dy.$$

We know that for any $y \ne 0$,

$$\lim_{k\to\infty} \left|\frac{\sin y/k}{y/k}\right| = 1.$$

Also, for any $B > 0$, we have

$$\left|\frac{\sin y}{y}\frac{\sin y/k}{y/k}\right| \chi_{[0,B]} \le \chi_{[0,B]} \in L^1(\mathbb{R}).$$

By the Dominated Convergence Theorem, we obtain

$$\lim_{k\to\infty} \frac{4}{\pi} \int_0^\infty \left|\frac{\sin y}{y}\right| \left|\frac{\sin(y/k)}{y/k}\right| \chi_{[0,B]} dy = \frac{4}{\pi} \int_0^\infty \left|\frac{\sin y}{y}\right| \chi_{[0,B]} dy.$$

This leads to the conclusion that for all $B > 0$, we have

$$\lim_{k\to\infty} ||f_k^\vee||_1 \geq \frac{4}{\pi} \int_0^B \left|\frac{\sin y}{y}\right| dy.$$

However, as shown in Exercise 2.59a (which was on last semester's final exam), we know that $\int_0^\infty |y^{-1} \sin y| dy = \infty$. Hence the right hand side can be made arbitrarily large, and $||f_k^\vee||_1 \to \infty$ as $k \to \infty$.

(c) Suppose $\mathcal{F} : L^1 \to C_0$ is onto. By Corollary 8.27, we conclude that $\mathcal{F} : L^1 \to C_0$ is therefore a bijection. By the Open Mapping Theorem, $\mathcal{F}^{-1} : C_0 \to L^1$ is a bounded linear operator. In other words, there exists a $C > 0$ such that for all $f \in C_0$, we have

$$||f^\vee||_1 \leq C||f||_u.$$

However, $||f_k||_u = 2$ for all positive integers k, but $||f_k^\vee||_1$ can be made arbitrarily large as $k \to \infty$. This contradiction establishes that $\mathcal{F}(L^1)$ is a proper subset of C_0.

Problem 8.26

The aim of this exercise is to show that the inverse Fourier transform of $e^{-2\pi|\xi|}$ on $\mathbb{R}^n$ is

$$\phi(x) = \frac{\Gamma(\frac{1}{2}(n+1))}{\pi^{(n+1)/2}} (1 + |x|^2)^{-(n+1)/2}.$$

a. *If $\beta \geq 0$, $e^{-\beta} = \pi^{-1} \int_{-\infty}^\infty (1+t^2)^{-1} e^{-i\beta t} dt$. (Use (8.37).)*

b. *If $\beta \geq 0$, $e^{-\beta} = \int_0^\infty (\pi s)^{-1/2} e^{-s} e^{-\beta^2/4s} ds$. (Use (a), Proposition 8.24, and the formula $(1+t^2)^{-1} = \int_0^\infty e^{-(1+t^2)s} ds$.*

c. *Let $\beta = 2\pi|\xi|$ where $\xi \in \mathbb{R}^n$; then the formula in (b) expresses $e^{-2\pi|\xi|}$ as a superposition of dilated Gauss kernels. Use Proposition 8.24 again to derive the asserted formula for ϕ.*

Solution:

(a) By (8.37), we have

$$e^{-2\pi|\xi|} = \mathcal{F}\left(\frac{1}{\pi(1+x^2)}\right) = \int_{-\infty}^\infty \pi^{-1}(1+x^2)^{-1} e^{-i2\pi x\xi} dx.$$

If we let $\xi = (2\pi)^{-1}\beta$, we obtain

$$e^{-\beta} = \pi^{-1} \int_{-\infty}^\infty (1+t^2)^{-1} e^{-i\beta t} dt.$$

(b) To justify using Fubini's Theorem later on, we first of all compute:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}|e^{-s}e^{-st^2}e^{-i\beta t}|dsdt = \int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-s(1+t^2)}dsdt = \int_{-\infty}^{\infty}\frac{dt}{1+t^2} = \Big|_{-\infty}^{\infty}\arctan t = \pi < \infty.$$

We now compute from (a) and the formula $(1+t^2)^{-1} = \int_0^\infty e^{-(1+t^2)s}ds$,

$$e^{-\beta} = \frac{1}{\pi}\int_{-\infty}^{\infty}(1+t^2)^{-1}e^{-i\beta t}dt = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-(1+t^2)s}e^{-i\beta t}dsdt.$$

By Fubini's Theorem, we can swap the order of the integrals,

$$e^{-\beta} = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}e^{-s}e^{-st^2}e^{-i\beta t}dtds.$$

We now substitute $z = \beta(2\pi)^{-1}t$ and obtain

$$e^{-\beta} = \frac{1}{\pi}\int_{0}^{\infty}e^{-s}\left(\int_{-\infty}^{\infty}\frac{2\pi}{\beta}e^{-\pi(4\pi s/\beta^2)z^2}e^{-2\pi iz}dz\right)ds = \frac{2}{\beta}\int_{0}^{\infty}e^{-s}\mathcal{F}(e^{-\pi(4\pi s/\beta^2)z^2})(1)\ ds.$$

The Fourier transform of the Gaussian can be computed using Proposition 8.24, which yields

$$e^{-\beta} = \frac{2}{\beta}\int_{0}^{\infty}e^{-s}\left(\frac{\beta^2}{4\pi s}\right)^{1/2}e^{-\beta^2/4s}ds = \int_{0}^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\beta^2/4s}ds.$$

(c) As done in (b), before lauching into the proof, we compute an integral that will later allow us to use Fubini's Theorem. The Gaussian is integrated using Proposition 2.53:

$$\begin{aligned}\int_{0}^{\infty}\int_{-\infty}^{\infty}\left|(\pi s)^{-1/2}e^{-s}e^{-\pi^2|\xi|^2/s}e^{2\pi ix\cdot\xi}\right|d\xi ds &= \int_{0}^{\infty}\int_{-\infty}^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\pi^2|\xi|^2/s}d\xi ds\\ &= \int_{0}^{\infty}(\pi s)^{-1/2}e^{-s}\left(\frac{\pi}{\pi^2/s}\right)^{n/2}ds\\ &= \frac{1}{\pi^{(n+1)/2}}\int_{0}^{\infty}s^{n/2+1/2-1}e^{-s}ds\\ &= \frac{1}{\pi^{(n+1)/2}}\Gamma(n/2+1/2) < \infty.\end{aligned}$$

We now prove the claim. By part (b), letting $\beta = 2\pi|\xi|$ we obtain

$$e^{-2\pi|\xi|} = \int_{0}^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\pi^2|\xi|^2/s}ds.$$

We use this to write

$$(e^{-2\pi|\xi|})^\vee(x) = \int_{-\infty}^{\infty} e^{-2\pi|\xi|}e^{2\pi i x\cdot\xi}d\xi = \int_{-\infty}^{\infty}\int_0^{\infty} (\pi s)^{-1/2}e^{-s}e^{-\pi^2|\xi|^2/s}e^{2\pi i x\cdot\xi}dsd\xi.$$

Applying Fubini, we exchange the order of integration

$$(e^{-2\pi|\xi|})^\vee(x) = \int_0^{\infty} (\pi s)^{-1/2}e^{-s}\int_{-\infty}^{\infty} e^{-\pi^2|\xi|^2/s}e^{2\pi i x\cdot\xi}\ d\xi ds = \int_0^{\infty}(\pi s)^{-1/2}e^{-s}\cdot(e^{-\pi^2|\xi|^2/s})^\vee(x)\ ds.$$

Proposition 8.24 gives us the inverse Fourier transform of a Gaussian, hence

$$(e^{-2\pi|\xi|})^\vee(x) = \int_0^{\infty}(\pi s)^{-1/2}e^{-s}\left(\frac{s}{\pi}\right)^{n/2}e^{-s|x|^2}ds = \frac{1}{\pi^{(n+1)/2}}\int_0^{\infty} s^{(n-1)/2}e^{-s(1+|x|^2)}ds.$$

We substitute $z = s(1+|x|^2)$ and obtain

$$(e^{-2\pi|\xi|})^\vee(x) = \frac{1}{\pi^{(n+1)/2}}\left(\frac{1}{1+|x|^2}\right)^{(n+1)/2}\int_0^{\infty} z^{\frac{1}{2}(n+1)-1}e^{-z}dz = \frac{\Gamma(\frac{1}{2}(n+1))}{\pi^{(n+1)/2}}(1+|x|^2)^{-(n+1)/2}.$$

Problem 8.31
Suppose $a > 0$. Use (8.37) to show that

$$\sum_{-\infty}^{\infty}\frac{1}{k^2+a^2} = \frac{\pi}{a}\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}.$$

Then subtract a^{-2} from both sides and let $a \to 0$ to show that $\sum_1^\infty k^{-2} = \pi^2/6$.

Solution:
Using the notation ϕ from (8.37), we have

$$\sum_{-\infty}^{\infty}\frac{1}{k^2+a^2} = \frac{\pi}{a^2}\sum_{-\infty}^{\infty}\frac{1}{1+(\frac{k}{a})^2} = \frac{\pi}{a^2}\sum_{-\infty}^{\infty}\ \phi(k/a).$$

By the Poisson summation formula, we obtain

$$\frac{\pi}{a^2}\sum_{-\infty}^{\infty}\ \phi(k/a) = \frac{\pi}{a^2}\sum_{-\infty}^{\infty}\ (\phi(k/a))^\wedge.$$

Using Theorem 8.22b, we have

$$\frac{\pi}{a^2}\sum_{-\infty}^{\infty}\ (\phi(k/a))^\wedge = \frac{\pi}{a^2}\sum_{-\infty}^{\infty}\ a\hat{\phi}(ka).$$

We now apply (8.37) to substitute for $\hat{\phi}$:

$$\begin{aligned}
\sum_{-\infty}^{\infty} \frac{1}{k^2+a^2} &= \frac{\pi}{a} \sum_{-\infty}^{\infty} \hat{\phi}(ka) \\
&= \frac{\pi}{a} \sum_{-\infty}^{\infty} e^{-2\pi|ka|} \\
&= \frac{\pi}{a} \left(1 + 2\sum_{k=1}^{\infty} \left(e^{-2\pi a}\right)^k\right) \\
&= \frac{\pi}{a} \left(1 + 2\left(\frac{1}{1-e^{-2\pi a}} - 1\right)\right) \\
&= \frac{\pi}{a} \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}.
\end{aligned}$$

Now that we have shown the desired identity, we subtract a^{-2} from both sides:

$$\sum_{-\infty}^{-1} \frac{1}{k^2+a^2} + \sum_{1}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi}{a} \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} - \frac{1}{a^2}.$$

After simplification, we obtain

$$2\sum_{k=1}^{\infty} \frac{1}{k^2+a^2} = \frac{a\pi - 1 + e^{-2\pi a}(1+a\pi)}{a^2(1-e^{-2\pi a})}.$$

Letting $a \to 0$, the left hand side goes to $2\sum_1^\infty k^{-2}$. For the right hand side, we repeatedly apply l'Hopital's rule (three times in total):

$$\begin{aligned}
\lim_{a\to 0} \frac{a\pi - 1 + e^{-2\pi a}(1+a\pi)}{a^2(1-e^{-2\pi a})} &= \lim_{a\to 0} \frac{\pi + \pi e^{-2\pi a} - 2\pi e^{-2\pi a}(1+a\pi)}{2a(1-e^{-2\pi a}) + 2\pi a^2 e^{-2\pi a}} \\
&= \lim_{a\to 0} \frac{4a\pi^3 e^{-2\pi a}}{2 - 2e^{-2\pi a} + 8a\pi e^{-2\pi a} - 4\pi^2 a^2 e^{-2\pi a}} \\
&= \lim_{a\to 0} \frac{4\pi^3 e^{-2\pi a} - 8a\pi^4 e^{-2\pi a}}{12\pi e^{-2\pi a} - 24\pi^2 a e^{-2\pi a} + 8\pi^3 a^2 e^{-2\pi a}} \\
&= \frac{4\pi^3}{12\pi}.
\end{aligned}$$

Therefore $2\sum_1^\infty k^{-2} = \pi^2/3$, which completes the proof.

Problem 8.35
The purpose of this exercise is to show that the Fourier series of "most" continuous functions on $\mathbb{T}$ do not converge pointwise.

a. *Define $\phi_m(f) = S_m f(0)$. Then $\phi \in C(\mathbb{T})^*$ and $||\phi|| = ||D_m||_1$.*

b. *The set of all $f \in C(\mathbb{T})$ such that the sequence $\{S_m f(0)\}$ converges is meager in $C(\mathbb{T})$. (Use Exercise 34 and the uniform boundedness principle.)*

c. *There exists $f \in C(\mathbb{T})$ (in fact, a residual set of such f's) such that $\{S_m f(x)\}$ diverges for every x in a dense subset of $\mathbb{T}$. (The result of (b) holds if the point 0 is replaced by any other point in $\mathbb{T}$. Apply Exercise 40 in §5.3.*

Solution:

(a) We have

$$\phi_m(f) = S_m f(0) = \int_{\mathbb{T}} f(y) D_m(y) dy.$$

By linearity of the integral, ϕ_m acts linearly on $C(\mathbb{T})$, and

$$|\phi_m(f)| \leq \int_{\mathbb{T}} |f| \; |D_m| dy \leq ||f||_u \; ||D_m||_1.$$

Therefore, $\phi_m \in C(\mathbb{T})^*$. To show $||\phi|| = ||D_m||_1$, we apply the Riesz Representation Theorem for Radon measures. Indeed, since Lebesgue measure is a Radon measure and $D_m \in L^1$, we know $d\mu = D_m dy$ is a Radon measure. Hence we can rewrite

$$\phi_m(f) = \int_{\mathbb{T}} f d\mu.$$

By definition, $d|\mu| = |D_m| dy$, hence by the Riesz Representation Theorem (7.17):

$$||\phi_m|| = ||\mu|| = |\mu|(\mathbb{T}) = \int_{\mathbb{T}} |D_m| dy = ||D_m||_1.$$

(b) Proceed by contradiction. Suppose the set of all $f \in C(\mathbb{T})$ such that the sequence $\{\phi_m(f)\}$ converges is nonmeager in $C(\mathbb{T})$. Then $\sup_m |\phi_m(f)| < \infty$ for all f in some nonmeager subset of $C(\mathbb{T})$. By the uniform boundedness principle, $\sup_m ||\phi_m|| < \infty$. However, by Exercise 34, $||\phi_m|| = ||D_m||_1 \to \infty$ as $m \to \infty$. This is a contradiction, and hence the set of all $f \in C(\mathbb{T})$ such that the sequence $\{\phi_m(f)\}$ converges is meager in $C(\mathbb{T})$.

(c) Enumerate the rational $\{r_k\}$ inside $(0, 1)$. This yields a countable dense subset of $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$. Define

$$T_{jk}(f) = S_j f(r_k).$$

Then

$$T_{jk}(f) = \int_{\mathbb{T}} f(y) D_j(r_k - y) dy = \int_{\mathbb{T}} f(y) \tau_{-r_k} D_j(-y) dy.$$

Denote $D_j(-y) = \tilde{D}_j$. We can repeat the same analysis done in (a) and obtain $T_{jk} \in C(\mathbb{T})^*$ and

$$||T_{jk}|| = ||\tau_{-r_k}\tilde{D}_j||_1 = ||D_j||_1$$

.

By the same argument of part (b), we conclude that for all r_k, the set of all $f \in C(\mathbb{T})$ such that $\sup_j |T_{jk}f| < \infty$ is meager in $C(\mathbb{T})$.

We now apply the principle of condensation of singularities (Exercise 5.40). We have shown that for each k there exists a $f \in C(\mathbb{T})$ (indeed, a residual set of f's) such that $\sup\{|T_{jk}f| : j \in \mathbb{N}\} = \infty$. By the principle of condensation of singularities, there is a residual set of f's in $C(\mathbb{T})$ such that $\sup\{|T_{jk}f| : j \in \mathbb{N}\} = \infty$ for all k. Hence we obtain we a residual set of $f \in C(\mathbb{T})$ such that $\{S_m f(r_k)\}$ diverges for each r_k.